AD-A038 294 BALLISTIC RESEARCH LABS ABERDEEN PROVING GROUND MD F/G 20/1

ROBUST IDENTIFICATION OF LINEAR SYSTEMS.(U)

FEB 77 V D VANDELINDE

UNCLASSIFIED BRL-1960 NL

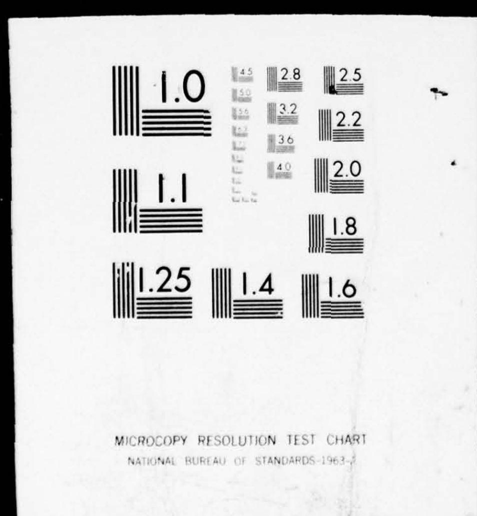
1 OF 3

AD A038294

SIFIED

1 OF 3

AD A038294



ADA038294

BRL R 1960

BRL

(12)

AD

REPORT NO. 1960

ROBUST IDENTIFICATION OF LINEAR SYSTEMS

V. David VandeLinde

February 1977

Approved for public release; distribution unlimited.

USA BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

(14) BRL [redacted]-1960

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

(6) ROBUST IDENTIFICATION OF LINEAR SYSTEMS.

5. TYPE OF REPORT & PERIOD COVERED

(9) Final rept.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

(10) V. David VandeLinde
Department of Electrical Engineering
The Johns Hopkins University

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS

US Army Ballistic Research Laboratory
Aberdeen Proving Ground, Maryland 21005

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

(16) RDTE Project 1T161101A91A

11. CONTROLLING OFFICE NAME AND ADDRESS

US Army Materiel Development & Readiness Command
5001 Eisenhower Avenue
Alexandria, Virginia 22335

12. REPORT DATE

(11) FEB[redacted] 1977

13. NUMBER OF PAGES

255

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

(12) 254p.

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Work supported by US Army Ballistic Research Laboratory through the Laboratory Research Cooperative Program of the Army Research Office, under Task Order 76-20, Basic Agreement DAHC04-72-A-0001.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Robust Estimation	Quantization
System Identification	Linear-Quadratic Estimation and Control
Robust Statistical Procedures	Cross-Correlation Methods
Linear Systems	

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) ams/kst

The first two sections of this report survey the current techniques of identification and time series analysis for linear, time invariant, single-input, single-output systems with noisy measurements. Typically, this additive noise has been modeled either as being completely specified (e.g., a Gaussian with known parameters) or as being required to satisfy general conditions such as "whiteness" and non-correlation with the input. Departures from the assumed noise model sometimes cause severe deterioration in the efficiency of

next page

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

050 750

20. ABSTRACT (Continued)

W sub k

cont. available identification algorithms. Since it seems reasonable to have some, if not complete, knowledge of the operating environment, it is assumed in this report that the measurement noise w_k has a distribution $F(w) = (1-\varepsilon)K(w) + \varepsilon C(w)$, where $K(\cdot)$ is a completely specified distribution and $C(\cdot)$ belongs to some broad class of distribution.

In the third section, a robust scheme for estimating the system cross correlations is proposed in order to desensitize the performance of the identification algorithm to the distribution of w_k. Extensive computer simulations show that the proposal provides a robust identification technique which has good uniform behavior over a variety of distributions for w_k.

TABLE OF CONTENTS

SECTION 1

A SURVEY OF IDENTIFICATION TECHNIQUES FOR SINGLE-INPUT SINGLE-OUTPUT, LINEAR, TIME INVARIANT, NOISY SYSTEMS MODELED IN DISCRETE TIME

Chapter I - General Introduction and Scope of the Report

The problem of identification arises as a natural consequence of control engineering, since any control system design requires knowledge of the parameters of the system to be controlled. The field is quite vast at present, so that a critical review of state-of-the-art techniques must of necessity be focussed on a specific problem. The scope of this discussion will be limited principally to popular identification methods for linear, time invariant, noisy systems modeled in discrete time. This is not as much of a restriction as one might be inclined to think, since the bulk of the research in identification has been in this area. We consider the discrete time formulation principally for ease in handling white noise, the modelling of which in continuous time is a technically difficult problem. Another good reason is that such models are naturally suited for digital computer simulation and have found wide usage.

There are several books and survey papers in the field. Eykhoff [1974] is the most current and most comprehensive text available at present; one may also refer to Sage and Melsa [1971] and Graupe [1972]. For survey papers, Åström and Eykhoff [1971] and Nieman, Fisher and Seborg [1971] may be consulted. The latter has an extensive bibliography. Proceedings of the 3rd IFAC Symposium on Identification and System Parameter Estimation [1973] and the special issue of the IEEE Transactions in Automatic Control [Dec. 1974] are quite

comprehensive and up-to-date in the coverage of the different aspects of identification. Quite a few of the papers serve the dual purpose of providing concise surveys and highlighting/presenting new results. Eykhoff [1974], Åström and Eykhoff [1971] and the special IEEE issue [Dec. 1974] have been used extensively in preparing this paper.

In the manner of Åström and Eykhoff [1971], the definition of identification given by Zadeh [1962] may well be chosen as our starting point:

"Identification is the determination, on the basis of input and output, of a system within a specified class of systems, to which the system under test is equivalent". Identification is thus the 'inverse problem' of system analysis; given an input and output-time history, determine the equations that describe the behavior of the system. Using Zadeh's definition, an identification problem is thus characterized by three quantities: a class of systems, $S = \{s\}$, a class of input signals, U, and a criterion to determine equivalence. The system under test will be referred to as the <u>process</u> and the elements of S will be called <u>models</u>. From the practical viewpoint, we believe it is important to consider a fourth characteristic of any identification technique, viz. its implementation - on-line or off-line. We shall consider briefly the basic elements of the identification problem: viz., the class of models, the class of input signals, the equivalence

criterion and the type of implementation. We shall then proceed to the task of discussing off-line and on-line techniques of identification for linear time invariant systems in stochastic environments.

Chapter II - Basic Elements of the Identification Problem

A. Class of Models

Any identification problem is basically formulated by the choice of model structure. The choice will depend not only on the a priori knowledge available but also on the purpose of the identification. It will greatly influence the nature of the identification problem, such as: the manner of utilizing the results of the identification in subsequent operations, the effort expended in computation, the possibility of obtaining unique solutions etc. There are, unfortunately, few general results available with regard to choice of structure.

The principal choice seems to be between parametric and nonparametric models. It might be instructive to consider the discussion by McGhee [1963], wherein he coins the terms 'function space description' and 'parameter space description'. The first uses the idea of tranformation defined over a function space. The function space provides a representation of the process input signal. Examples of such representations can be found in the Fourier series expansion, the Laguerre function expansion etc. The transformation that is defined over this space follows from the dynamics of the process. The process output signal may be represented on a similar space. In these terms the identification problem is to find what transformation from input function space to output function space characterizes the process. As no information about the physical structure of the process or its assumed mathematical equations is used,

this approach is of the 'black box' identification type.

The 'parameter space description' starts from an assumed mathematical description of the process dynamics. This description is a parametric model of finite dimension. The coordinates of the parameter space are the numerical values of the quantities that determine the 'output' of the model. If, for example, the assumed description is an ordinary differential equation, then the coordinates may be the values of the coefficients and the initial conditions. If there is no forcing funtion (input), then from this one point in the parameter space one can predict the process output. If there is a forcing function, then the unknown parameters of that signal increase the dimensionality of the parameter space. The dimensionality remains finite, while in principle an infinite number of parameters has to be determined in the function space description. Consequently one distinguishes between:

Nonparametric models

e.g. a) impulse responses

b) transfer functions

c) spectral densities

and

Parametric models

e.g. a) differential (difference) equations of predetermined form and order

b) state models

It is known that parametric models can give results with large errors if the order of the model does not agree with the

order of the process. Nonparametric representations have the advantage that it is not necessary to specify the order of the process explicitly. These representations are, however, intrinsically infinite dimensional.

For linear systems, in order to obtain unique solutions as well as to be able to construct efficient algorithms it is of interest to find representations of the system which contain the smallest number of parameters, viz. CANONICAL representations. A comprehensive up-to-date study of canonical forms will be found in Denham [1974] which contains also the essential contributions of Popov [1972], Weinert and Anton [1972], Caines [1972], Mayne [1972 a & b] and Kalman [1973]. Also, Tse and Weinert [1973] have reported a scheme for determining system order, whereas tests of order for parametric models have been proposed by Anderson [1962], Åström [1968] and Woodside [1970]. Dickinson, Kailath and Morf [1974] expose the interrelations between frequency domain and state space descriptions of multivariable linear systems.

In the case of linear systems, the knowledge of one characteristic process time function (e.g. impulse response) is sufficient to determine process output for arbitrary input signals. Such a procedure has advantages which are also desirable for the description of nonlinear processes (George [1959]), viz.

a) it gives an explicit input/output relationship

b) it facilitates the discussion of combinations of systems

c) it allows the consideration of random inputs

For some classes of nonlinear processes these requirements are fulfilled by Volterra series. The Volterra [1959] series method treats the linear case as a sub-case of the nonlinear case, a very desirable property indeed. Flake [1963], Schetzen [1974], Baumgartner and Rugh [1975] and Harper and Rugh [1975] have developed methods for identifying certain classes of nonlinear processes using Volterra series.

As a concluding note on process models, it may be worthwhile to mention the question of identifiability. This has been surveyed by Glover and Willems [1974] and been discussed by Tse, Weinert, Anton and Mehra [1973], Balakrishnan [1969] and Staley and Yue [1970].

B. Class of Inputs

It is well known (and as may be reasonably expected) that significant simplifications in computations can be achieved by choosing input signals of a special type, e.g. impulse functions, step functions, 'colored' or white noise, sinusoidal signals, pseudo random binary sequences (PRBS) etc. A bibliography on PRBS is given in Nikiforuk and Gupta [1969]. Refer also to Godfrey [1970]. For the use of deterministic signals see Strobel [1968], VandenBos [1970], Welfonder and Hasenkopf [1970], Cumming [1970]. For periodic test signals, see VandenBos [1974].

From the viewpoint of applications it would be very desirable to use techniques which do not make strict limitations

on the inputs. However, if the input signals can be chosen, how should this be done? The problem of designing good probing or test signals is an important one in many industrial applications and Goodwin, Zarrop and Payne [1974] discuss the simultaneous design of test signals, sampling intervals and input filters. A survey of optimal input synthesis is done by Mehra [1974] who has also obtained new results using the statistical studies of Kiefer and Wolfwowitz [1959] on experimental design for regression problems. Keviczky [1975] works along similar lines in 'designing' optimal inputs for identification as extensions of the work of Box and Draper [1971], Kiefer [1961] and Wynn [1970].

C. Criterion for Equivalence

The criterion for determining equivalence in the definition of the identification problem is often a minimization of a scalar loss function. Mostly, the criterion is expressed as a functional of an error, e.g.

$$V(y, y_M) = \int_0^T e^2(t)\,dt \tag{1}$$

where y is the process output, y_M the model output and e the error; y, y_M and e are considered functions defined on $(0,T)$. In the case

$$e = y - y_M = y - M(u) \tag{2}$$

where $M(u)$ denotes the output of the model when the input is u, e is called the output error (Fig. 1).

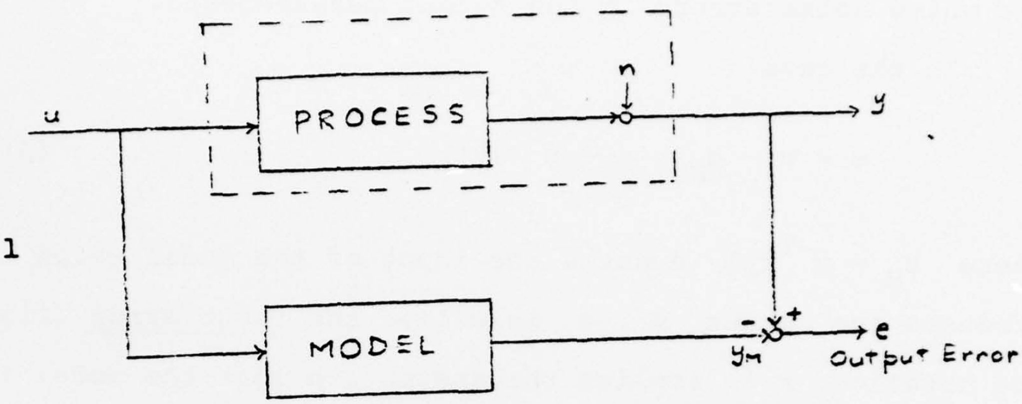


Fig. 1

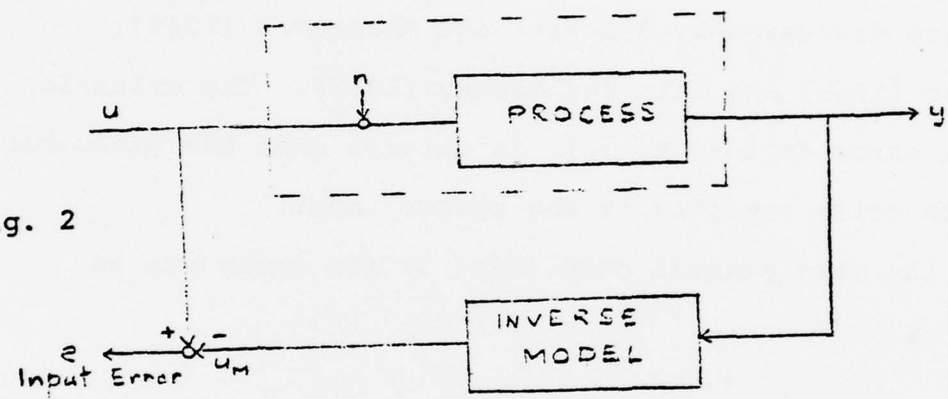


Fig. 2

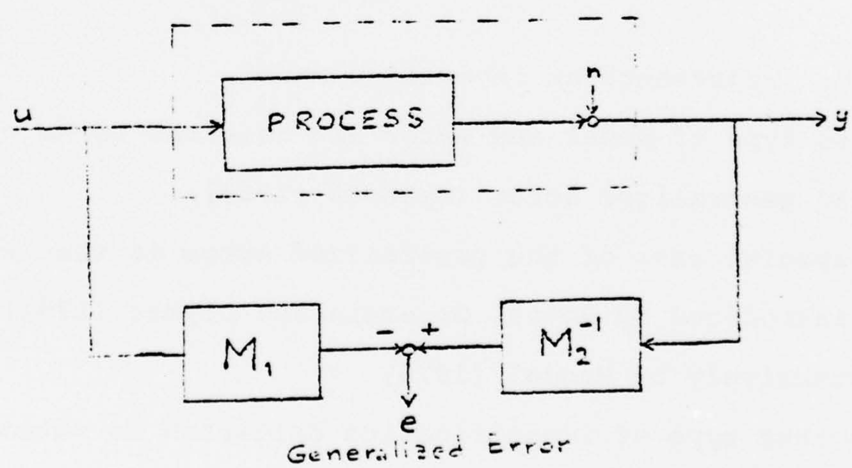


Fig. 3

It is the natural definition when the only disturbances are white noise errors in the output measurements.

In the case

$$e = u - u_M = u - M^{-1}(y) \tag{3}$$

where $u_M = M^{-1}(y)$ denotes the input of the model which produces the output y, e is called the <u>input error</u> (Fig. 2). The notation M^{-1} implies the assumption that the model is invertible. Rigorous definitions of the concept of invertibility are discussed by Brockett and Mesarovic [1965], Silverman [1969] and Sain and Massey [1969]. The criterion (1) with error defined by (3) is natural when the disturbances are white noise entering at the system input.

In the more general case (Fig. 3) the error can be defined as

$$e = M_2^{-1}(y) - M_1(u) \tag{4}$$

where M_2 represents an invertible model.

This type of model and error are referred to as generalized model and generalized error (Eykhoff [1963]).

A special case of the generalized error is the "equation error" introduced by Potts, Ornstein and Clymer [1961], and used extensively by Mendel [1973].

Another type of identification criterion is obtained by imbedding the problem in a probabilistic framework. If S

is defined as a parametric class, $S = \{s_\beta\}$, where β is a parameter, the identification problem becomes a parameter estimation problem, enabling the use of the tools of estimation and decision theory. However, in many probabilistic situations it turns out that the estimation problem can be reduced to an optimization problem, with the loss function given by the probabilistic assumptions.

D. <u>Type of Implementation</u>
(Eykhoff [1974])

All solutions to parametric identification problems consist of finding the extremum of the loss functions considered as a function of the parameter β. A distinction then can be made with respect to the type of implementation.

Consider the correspondence between process and model to be established by the error criterion

$$V\{y, y_M; \beta\} = \int_0^T \{y(t) - y_M(t;\beta)\}^2 dt$$

$$\text{where } \beta = (\beta_1, \ldots, \beta_n)$$

Then one can follow one of two strategies:

I. Put $\frac{\partial V}{\partial \beta_i} \equiv 0$ for $i = 1, \ldots, n$.

This is a necessary condition for obtaining the minimal error. These n equations with n unknown estimates $(\beta_1, \ldots, \beta_n)$ can be solved for the β's, thus providing explicit mathematical relations to obtain numerical quantities.

II. Put $\frac{\partial V}{\partial \beta_i} \to 0$ for $i = 1, \ldots, n$.

A convergence towards zero can be obtained if $\frac{\partial V}{\partial \beta_i}$ can be determined by a suitable instrumentation and if these values are then used for the adjustment of a physical model such that model characteristics approach process characteristics in some pre-determined sense.

One can thus consider the type I approach as basically an OFF-LINE one-shot technique where the estimate (of β)

a) is available after a finite number of elementary operations

b) requires considerable memory

c) is not available in an approximate form as an intermediate result

d) is open loop with respect to the estimate.

On the other hand, the type II approach is an iterative ON-LINE procedure where the estimate

a) is available (in principle) after an infinite number of elementary operations

b) requires less memory

c) is available in an approximate form as an intermediate result

d) is closed loop with respect to the estimate

e) is found by a self correcting procedure.

In terms of engineering applications, on-line schemes have been favored heavily because of properties (b)-(e). However, on-line methods usually suffer from the drawback of just guessing initial estimates.

The rest of the paper will be devoted to brief descriptions of the popular identification techniques for discrete models of linear time invariant systems in stochastic environments. The format of presentation will be to first describe off-line techniques, followed by a listing of on-line methods. Quite often it will be seen that an off-line technique has been put into recursive form for on-line implementation. An effort has been made to state, for each technique, the necessary assumptions and the a priori knowledge required. Since it is only possible to identify the controllable and observable parts of a linear system from input/output data (Kalman [1963]), there is no loss of generality to assume that all processes of interest in this report are controllable and observable. In addition, the processes are assumed to be stable. Furthermore, we consider only single-input single-output systems for convenience of representation and in consideration of the fact that it is the most studied problem in system identification

Chapter III - Off-line Techniques for Identification

It is logical to start with the oldest technique - least squares identification of a parametric model.

A. Least Squares
(Isermann, Baur, Bamberger, Kneppo and Siebert ([1974])

Process Model:
(Fig. 4)

$$y(k)+\hat{a}_1 y(k-1)+\dots+\hat{a}_n y(k-n) = \hat{b}_1 u(k-1)+\dots+\hat{b}_n u(k-n)+e(k) \tag{6}$$

with $y \rightarrow$ Process output $\quad e \rightarrow$ Generalized error

$u \rightarrow$ Process input

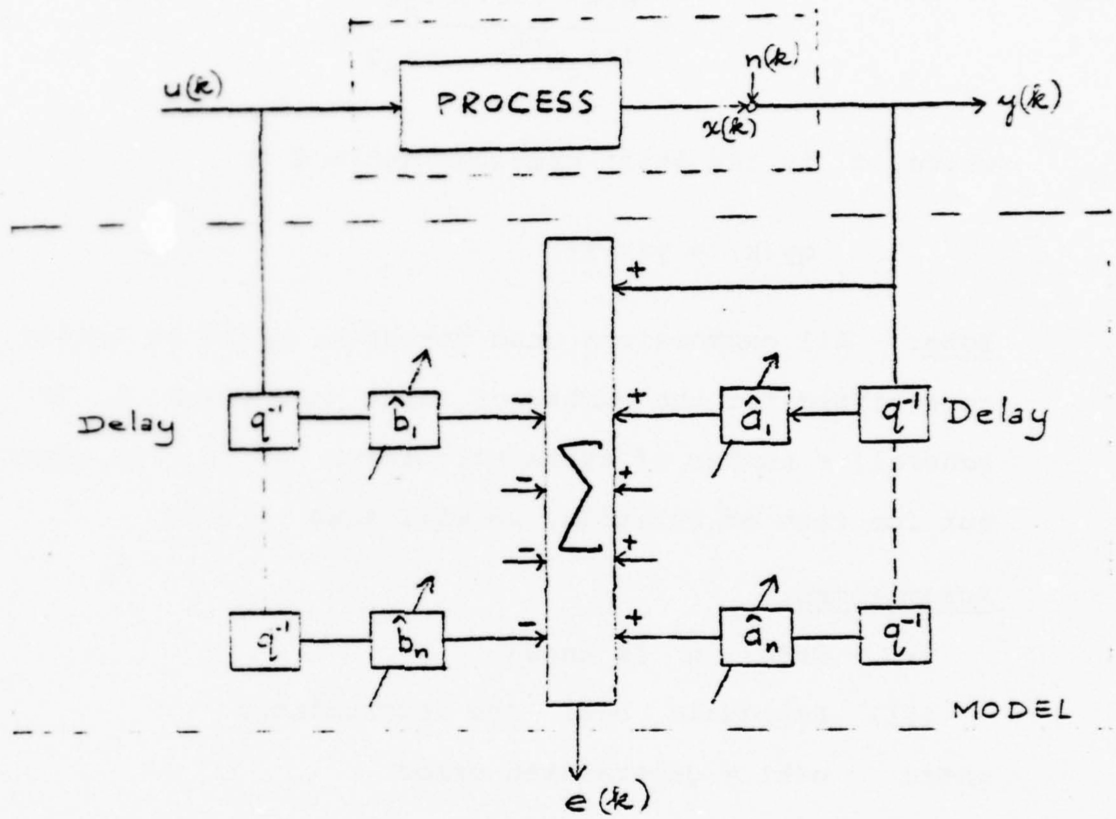
or $\quad y_k \hat{a} = u_k \hat{b} + e_k$ where $e_k = e(k)$ and

$$\begin{aligned} y_k &= [y(k) \quad y(k-1) \ \dots y(k-n)] \\ u_k &= [u(k-1) \ u(k-2) \ \dots u(k-n)] \\ \hat{a}^T &= [\ 1 \ \ \hat{a}_1 \ \dots\dots\dots\dots \hat{a}_n] \\ \hat{b}^T &= [\ \hat{b}_1 \ \hat{b}_2 \ \dots\dots\dots\dots \hat{b}_n] \end{aligned} \tag{7}$$

For the actual process, because of additive noise $n(k)$ in the process output $y(k)$, the equations are:

$$x(k)+a_1 x(k-1)+\dots+a_n x(k-n) = b_1 u(k-1)+\dots+b_n u(k-n)$$

$$y(k) = x(k)+n(k)$$



u(k)
PROCESS
n(k)
x(k)
y(k)
+
Delay
q^{-1}
$\hat{b}_1$
−
+
$\hat{a}_1$
q^{-1}
Delay
$\sum$
q^{-1}
$\hat{b}_n$
$\hat{a}_n$
q^{-1}
MODEL
e(k)

Fig. 4

The process may also be represented by the pulse transfer function

$$G(q^{-1}) = \frac{\underline{X}(q)}{U(q)} = \frac{B(q^{-1})}{A(q^{-1})} \qquad (8)$$

$$= \frac{b_1 q^{-1} + \ldots + b_n q^{-n}}{1 + a_1 q^{-1} + \ldots + a_n q^{-n}}$$

where q is the shift operator defined by

$$qy(k) = y(k+1) \qquad (9)$$

Note: All expressions used hereafter could be easily generalized for the number of input parameters b_i [m , in general] $\neq$ number of state parameters a_i [n , in general] , but for ease of notation, we will take $m = n$.

Assumptions:

(i) Order n is known

(ii) Residuals $e(k)$ are uncorrelated

where $e(k)$ = generalized error

$$= y(k) + \hat{a}_1 y(k-1) + \ldots + \hat{a}_n y(k-n) - \hat{b}_1 u(k-1) - \ldots - \hat{b}_n u(K-n) \qquad (10\text{-a})$$

$$= y(k) - y_M(k)$$

$$= y(k) - \psi(k)\hat{\beta}$$

where $y_M(k)$ is the prediction of the model based on process observations $y(k-n), \ldots, y(k-1)$

$$\psi(k) = [-y(k-1)\ldots.-y(k-n)\ldots u(k-1)\ldots.u(k-n)]$$

$$\hat{\beta}^T = [\hat{a}_1 \ldots.. \hat{a}_n \ldots \hat{b}_1 \ldots. \hat{b}_n]$$

Minimizing the loss function

$$V = \sum_{k=n}^{N+n} e^2(k) \tag{10-b}$$

and using the notation

$$y^T = [y(n)y(n+1)\ldots..y(n+N)]$$

$$\underline{\Psi} = \begin{bmatrix} -y(n-1) -y(n-2) \ldots -y(0) & u(n-1)u(n-2)\ldots u(0) \\ -y(n) \ldots\ldots\ldots -y(1) & u(n) \quad \vdots \\ \vdots \qquad\qquad \vdots & \vdots \\ -y(n+N-1)\ldots\ldots -y(N) & u(n+N-1)\ldots\ldots u(N) \end{bmatrix} \tag{11}$$

we note that e is linear in $\hat{a}_i$ and $\hat{b}_i$.

Thus V is minimized by

$$\hat{\beta}_{LS} = [\underline{\Psi}^T\underline{\Psi}]^{-1}\underline{\Psi}^T y \qquad \text{if } \underline{\Psi}^T\underline{\Psi} \text{ is not singular.}$$

Thus the least squares estimate is simply calculated from a set of observations y and a set of inputs u.

<u>Source(s) of error</u>:

(i) If assumed order is wrong, there can be considerable error. So test for model order

(ii) Principal problem is of correlated $e(k)$, as is quite usually the case.

$\hat{\beta}_{LS}$ can still be computed but suffers from bias as can be seen simply by examining $\underline{\Psi}$. The y elements of $\underline{\Psi}$ are really

$$y(k) = x(k) + n(k) \tag{13}$$

where $n(k)$ is additive measurement noise at the process output. The bias is caused by the $n^2(k)$ terms in $\underline{\Psi}^T\underline{\Psi}$ and its asymptotic value is

$$E[\hat{\beta}-\beta] = [E(\underline{\Psi}^T\underline{\Psi})]^{-1} E(\underline{\Psi}^T e) \tag{14}$$

where $\qquad \beta^T = [a_1 \ldots . a_n \ldots b_1 \ldots . b_n]$

To deal with correlated residuals several techniques have been suggested - generalized least squares, maximum likelihood, instrumental variables.

B. Generalized Least Squares
(Isermann et al [1974])

Clarke [1967] tries to overcome the bias problem by introducing filters.

Process Model:
(Fig. 5)

$$A_M(q^{-1})y(k) = B_M(q^{-1})u(k) + w(k) \tag{15}$$

or $\qquad y_k\hat{a} = u_k\hat{b} + w(k)$

where $w(k)$ are correlated random variables and the other quantities are defined as before.

(i) The first step is obtaining a L.S. estimate from (15), which result in biased estimates $\hat{\beta}_1$.

(ii) Then the residuals are calculated and analyzed by autoregression, assuming a model

$$w(k) = -f_1 w(k-1) - f_2 w(k-2) \ldots - f_\nu w(k-\nu) + e(k) \tag{16}$$

respectively. That is

$$w(k) = \xi(k) f + e(k) \tag{17}$$

where $e(k)$ are uncorrelated random variables and the order ν has to be chosen properly. Least squares estimation of the filter parameters using

$$\underset{w}{\begin{bmatrix} w(n) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w(n+N) \end{bmatrix}} = \underset{\underline{W}}{\begin{bmatrix} -w(n-1) \ldots\ldots -w(n-\nu) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -w(n+N-1) \ldots -w(n-\nu+N) \end{bmatrix}} \underset{f}{\begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_\nu \end{bmatrix}} + \underset{e}{\begin{bmatrix} e(n) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e(n+N) \end{bmatrix}} \tag{18}$$

leads to $f = [\underline{W}^T \underline{W}]^{-1} \underline{W}^T w$ (19)

(iii) The input and output sequences are filtered according to

$$\begin{aligned} \tilde{u}(k) &= u_k \hat{f} + u(k) \\ \tilde{y}(k) &= y_k \hat{f} + y(k) \end{aligned} \tag{20}$$

(iv) A new L.S. fit is made with these filtered $\tilde{u}(k)$ and $\tilde{y}(k)$ and new matrices $\underline{\Psi}$.

(v) Repeat from (ii).

This method is not iterative in the process input-output data; a whole sequence of observations is handled in a one-shot manner.

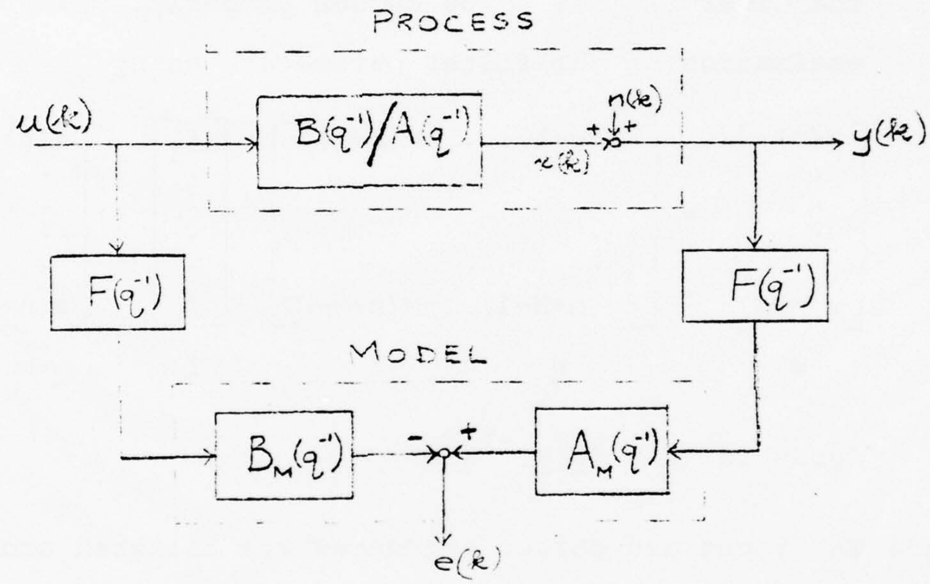


Fig. 5

Assumptions and associated sources of error:

(i) Order n is assumed known, hence the same error as was mentioned in L.S. case could occur.

(ii) Order ν of the filter has to be chosen appropriately or $\hat{\beta}$ estimates will still be biased.

C. Maximum Likelihood
(Eykhoff [1974])

This is a widely used technique pioneered by Åström and Bohlin [1965]. Up-to-date discussions may be found in Åström and Söderström [1974], Gupta and Mehra [1974] and Kashyap and Nasburg [1974].

Process Model:
(Fig. 6)

Taking the process as

$$A(q^{-1})y(k) = B(q^{-1})u(k) + \lambda C(q^{-1})n(k) \qquad (21)$$

where all quantities are defined as before and

$$n(k) = \text{additive noise, independent, } N(0,1)$$
$$\lambda = \text{level of noise signal}$$

$$C(q^{-1}) = 1+c_1q^{-1}+c_2q^{-2} +\ldots+ c_nq^{-n}$$

one may choose for the model

$$C_M(q^{-1})e(k) = A_M(q^{-1})y(k)-B_M(q^{-1})u(k) \qquad (22)$$

Written in vector form, (21) and (22) become

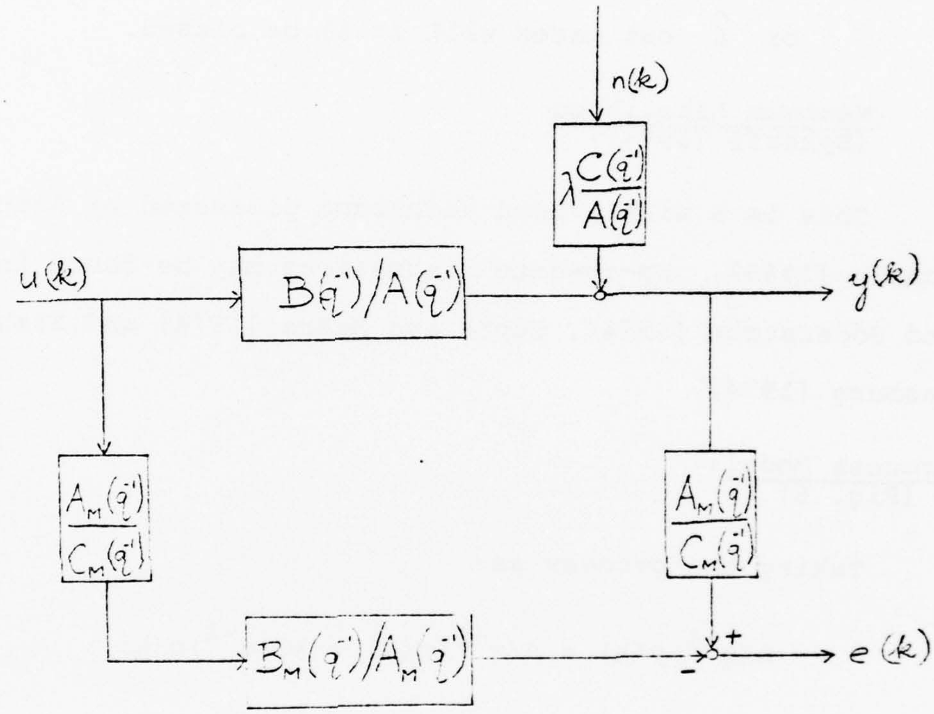

n(k)
$\lambda \frac{C(q^{-1})}{A(q^{-1})}$
u(k)
$B(q^{-1})/A(q^{-1})$
y(k)
$\frac{A_M(q^{-1})}{C_M(q^{-1})}$
$\frac{A_M(q^{-1})}{C_M(q^{-1})}$
$B_M(q^{-1})/A_M(q^{-1})$
+
−
e(k)

Fig. 6

$$\underline{a}y = \underline{b}u + \lambda\underline{c}\; n \tag{21-a}$$

$$\underline{\alpha}y = \underline{\beta}u + \gamma e \tag{22-a}$$

Because the noise n is Gaussian, the likelihood function for $e^T = \{e(1),\ldots,e(N)\}$ can be found as:

$$L(e;u,\alpha,\beta,\gamma,\lambda) = \frac{1}{(2\pi)^{N/2}\lambda^N} \exp\left[\frac{-e^Te}{2\lambda^2}\right] \tag{23}$$

$$\ell n\, L = -\frac{N}{2}\ell n\, 2\pi - N\ell n\,\lambda - \frac{1}{2\lambda^2}\sum_{k=1}^{N} e^2(k) \tag{24}$$

The M.L. estimate $\hat{\lambda}$ follows from

$$\frac{\partial}{\partial\lambda}\ell n\, L = \left[-\frac{N}{\lambda} + \frac{\sum_{k=1}^{N} e^2(k)}{\lambda^3}\right]\Bigg|_{\lambda=\hat{\lambda}} = 0 \tag{25}$$

or $$\hat{\lambda}^2 = \frac{\sum_{k=1}^{N} e^2(k)}{N}$$

The likelihood function may be considered as a function θ and λ, where $\theta^T = [\hat{a}_1\ldots.\hat{a}_n \vdots \hat{b}_1\ldots\hat{b}_n \vdots \hat{c}_1\ldots\hat{c}_n]$. The logarithm of the likelihood function may be noted to be linear in $\hat{a}_i$ and $\hat{b}_i$ and non-linear in $\hat{c}_i$. Consequently, finding the maximum of L or $\ell n\, L$ by differentiation and equating to zero is not necessarily a simple procedure. One may proceed as follows. First determine θ such that

$$V(\theta) = \sum_{k=1}^{N} e^2(k) \tag{26}$$

is minimal with respect to θ.

Then, $\hat{\lambda}^2 = \frac{1}{N} \min_{\theta} V(\theta)$ (27)

The popularity of ML estimates stems principally from their properties of consistency, asymptotic efficiency and asymptotic normality. It is also possible to theoretically compute the accuracy of the estimates.

Assumptions and sources of error:

(i) Order n is assumed known - hence the possibility of the associated error.

(ii) Normal distribution is assumed for $n(k)$. Non-normality may seriously affect the desirable properties of ML estimates (viz. consistency, asymptotic efficiency, etc.)

D. Instrumental Variabales (Isermann et al. [1974])

Generalized least squares and maximum likelihood methods use as their noise model a filter driven by white noise. If only the process dynamics are of interest then instrumental variables can be used to deal with correlated residuals. Refer to Wong and Polak [1967], Young [1970], Finigan and Rowe [1974] and Pandya [1974].

Process Model:

$$y = \underline{\Psi}\hat{\beta} + e \qquad (28)$$

with $e^T = [e(n)\ e(n+1)\ .....\ e(n+N)]$ and the other quantities as in equation (11) for the least squares case.

Premultiplying (28) by J^T so that

$$J^T y = J^T \underline{\Psi}\hat{\beta} + J^T e \tag{29}$$

where J is called the instrumental matrix satisfying

$$\left.\begin{aligned} &E\{J^T e\} = 0 \\ &E\{J^T \underline{\Psi}\} \text{ nonsingular} \end{aligned}\right. \tag{30}$$

the parameter estimates are then obtained as unbiased ones from

$$\hat{\beta} = [J^T \underline{\Psi}]^{-1} J^T y \tag{31}$$

The elements of J are chosen to be uncorrelated with the residuals e.

Wong and Polak [1967] and Young [1970] demonstrated the existence of optimal instrumental variables and they used the calculated, undisturbed output signal as instrumental variables, taking the parameter estimates as the parameters of an auxiliary model. If h is the output of the auxiliary model, the instrumental matrix becomes

$$J = \begin{bmatrix} -h(n-1) & \ldots\ldots & -h(0) . u(n-1) & \ldots\ldots & u(0) \\ \vdots & & \vdots \quad \vdots & & \vdots \\ -h(n+N-1) & & -h(N) . u(n+N-1) & & u(N) \end{bmatrix} \tag{32}$$

<u>Assumptions and sources of error:</u>

(i) System order n has to be assumed known - hence the usual chance of error.

(ii) Possibly, correlation between auxiliary model parameters

and e may not be zero, violating (30) and destroying the unbiasedness of the estimate.

This is a popular method which works well, the drawback being the added computation of the instrumental matrix.

As a final note on off-line techniques, we should mention that the identification scheme developed by Tse and Weinert [1973] using the canonical form of Weinert and Anton [1972] does not assume the order n of the process under investigation but estimates it from input/output data before estimating the parameters. The process model assumed is

$$\begin{aligned} x(k+1) &= Ax(k) + B\nu(k) \\ y(k) &= Cx(k) + \nu(k) \end{aligned} \tag{33}$$

where A, B, C are unknown matrices, and so are n (the process order) and Q, the covariance of the zero mean Gaussian process $\{\nu(k)\}$.

Chapter IV - On-line Techniques for Identification

We now consider identification technqiues with the widest application, viz. on-line schemes. As we stated earlier, several of the off-line technqiues can be put into iterative form (with respect to new measurements) for on-line implementation.

A. On-line Least Squares

Process model and assumptions:

These are identical to the off-line case. The recursive least squares estimate is obtained by writing equation (12)

$$\hat{\beta}_{LS} = [\underline{\psi}^T \underline{\psi}]^{-1} \underline{\psi}^T y$$

in partitioned form and introducing the matrix inversion lemma, Friedman [1954]:

$$\begin{aligned} \hat{\beta}(k+1) = \hat{\beta}(k) &+ [\psi(k+1)P(k)\psi^T(k+1)+1]^{-1} \cdot \\ &\cdot P(k)\psi^T(k+1)[y(k+1)-\psi(k+1)\hat{\beta}(k)] \end{aligned} \tag{34}$$

with

$$\begin{aligned} P(k+1) = P(k)[I-\psi^T(k+1)\psi(k+1)P(k) \cdot \\ \cdot [\psi(k+1)P(k)\psi^T(k+1)+1]^{-1}] \end{aligned} \tag{35}$$

$$P = [\underline{\psi}^T \underline{\psi}]^{-1}. \tag{36}$$

Sources of error:

Same as for the off-line case. Added problem of choosing $P(0)$ and $\hat{\beta}(0)$ which may be taken as 0.

B. On-line Generalized Least Squares (Isermann et. al [1974])

A recursive generalized least squares algorithm was developed by Hastings-James and Sage [1969]. Using the same process model and assumptions, the recursive equations are set up as in (34) and (35):

$$\hat{\beta}(k+1) = \hat{\beta}(k)+[\psi(k+1)\tilde{P}(k)\psi^T(k+1)+1]^{-1}\cdot \cdot P(k)\psi^T(k+1)[\tilde{y}(k+1)-\psi(k+1)\hat{\beta}(k)] \quad (37)$$

$$\tilde{P}(k+1) = \tilde{P}(k)[I-\psi^T(k+1)\psi(k+1)\tilde{P}(k)\cdot \cdot[\psi(k+1)\tilde{P}(k)\psi^T(k+1)+1]^{-1}] \quad (38)$$

$$\hat{f}(k+1) = \hat{f}(k)+[\xi(k+1)Q(k)\xi^T(k+1)+1]^{-1}\cdot \cdot Q(k)\xi^T(k+1)[w(k+1)-\xi(k+1)\hat{f}(k)] \quad (39)$$

$$Q(k+1) = Q(k)[I-\xi^T(k+1)\xi(k+1)Q(k)\cdot [\xi(k+1)Q(k)\xi^T(k+1)+1]^{-1}] \quad (40)$$

Initial matrices $\tilde{P}(0)$ and $Q(0)$ can be chosen as diagonal matrices with elements as large as possible without creating instability. The initial $\hat{\beta}(0)$ can be zero. An exponential weighting of past data using a weighting factor ρ (Hastings-James and Sage [1969]), in the terms

$$[\tilde{\Psi}(k+1)\tilde{P}(k)\tilde{\Psi}^T(k+1)+\rho] \quad \text{of eqs. (37),(38)}$$

$$\tilde{P}(k+1) = \frac{1}{\rho}\tilde{P}(k)[I - \ldots.] \quad \text{eq. (38)}$$

and in the analogous terms of (39) and (40), prevents the first estimates from becoming too poor, thus improving the convergence.

C. On-line Instrumental Variables

Process model and assumptions:

These are the same as for the off-line method. As in equations (34) and (35), one may write the recursive form for on-line implementation:

$$\hat{\beta}(k+1) = \hat{\beta}(k)+[\psi(k+1)P(k)j^T(k+1)+1]^{-1}\cdot \cdot P(k)j^T(k+1)[y(k+1)-\psi(k+1)\hat{\beta}(k)] \tag{41}$$

with

$$P(k+1) = P(k)[I-j^T(k+1)j(k+1)P(k)\cdot \cdot [\psi(k+1)P(k)j^T(k+1)+1]^{-1}] \tag{42}$$

$$P(k) = [\underline{J}^T(k)\underline{\Psi}(k)]^{-1} \tag{43}$$

$$j(k) = [-h(k-1) \;\ldots.\; -h(k-n)\,|\,u(k-1)\ldots.u(K-n)] \tag{44}$$

Young [1972] introduced a time delay and a low pass filter (Fig. 7) before updating the auxiliary model, to ensure that the auxiliary model parameters are not correlated with e at the same instant and to smooth the estimates. One may use a low pass filter $\hat{\beta}_{aux}(k) = (1-\gamma)\hat{\beta}_{aux}(k-1)+\gamma\hat{\beta}(k)$ where γ is small ($\gamma \simeq 0.03$).

Sources of error:

In addition to the possibilities of error mentioned for the off-line method, choice of γ influences the algorithm. $P(0)$ has to be chosen, and it may be taken as a diagonal matrix with large elements. Initial values of $\hat{\beta}$ and $\hat{\beta}_{aux}$ can be zero.

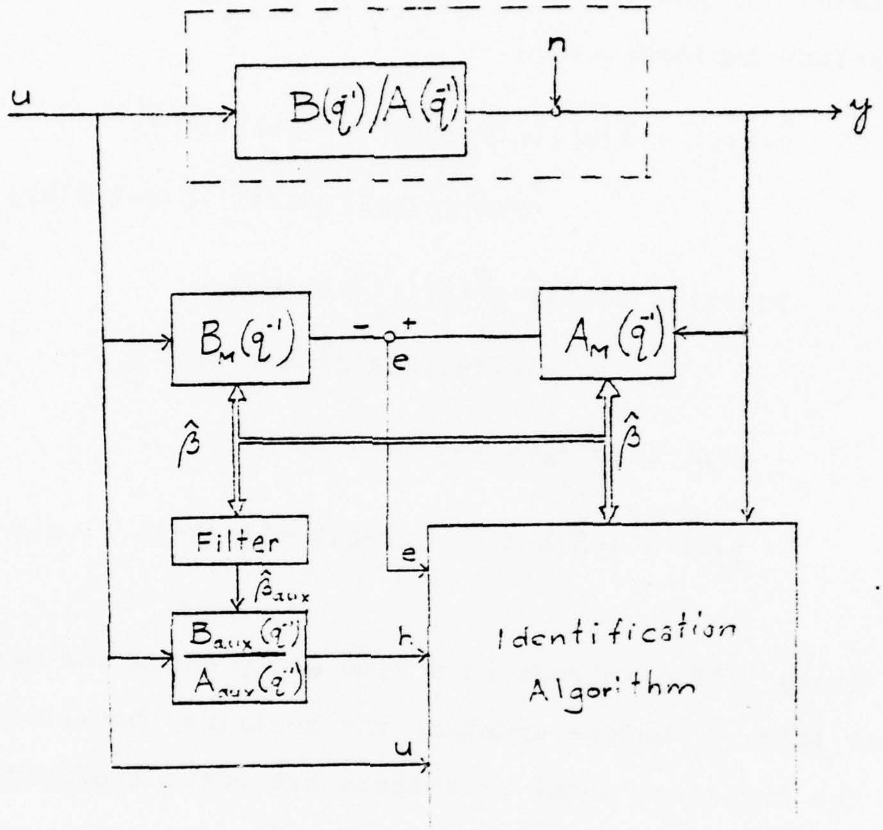

u
B(q⁻¹)/A(q⁻¹)
n
y
$B_M(q^{-1})$
$A_M(q^{-1})$
e
$\hat{\beta}$
$\hat{\beta}$
Filter
$\hat{\beta}_{aux}$
$\frac{B_{aux}(q^{-1})}{A_{aux}(q^{-1})}$
e
h
u
Identification
Algorithm

Fig. 7

D. Stochastic Approximation
(Isermann et. al [1974]

This is one of the most popular methods currently in use for sequential estimation of process parameters. It was first introduced by Robbins and Monro [1951], generalized by Dvoretsky [1956] and treated extensively by Albert and Gardner [1967]. Its main characteristic is the simplicity of its implementation which makes it attractive. An up-to-date survey is Saridis [1974 a].

Process model:

The same model as in the least squares case is used, i.e. single input-single output description as in (10-a)

$$y(k)+\hat{a}_1 y(k-1)+\ldots+\hat{a}_n y(k-n) = \hat{b}_1 u(k-1)\ldots+\hat{b}_n u(k-n)+e(k)$$

Consider the estimation of the impulse response, $g(\nu)$, of this process. The following algorithm may be used (to illustrate stochastic approximation).

$$\hat{G}(k+\ell) = \hat{G}(k-1)+\gamma(k)U(k+\ell)\cdot[y(k+\ell) -U^T(k+\ell)\hat{G}(k-1)] \tag{45}$$

with

$$\hat{G}^T = [\hat{g}_1,\hat{g}_2,\ldots,\hat{g}_\ell] \tag{46}$$

$$U^T(k) = [u(k-1)u(k-2)\ldots.u(k-\ell)] \tag{47}$$

$$\gamma(k) = \gamma(\xi) = \frac{1}{\xi(k)}\ ;\ \xi(k) = \frac{k-1}{\ell+1}$$

$$k = 1,\ell+2,2\ell+3,3\ell+4,\ldots. \tag{48}$$

If there are 2n unknown parameters of β, $\ell = 2n$ values of the g_i are estimated by equations (45) and the parameters $\hat{a}_i$ and $\hat{b}_i$ are calculated as follows.

Since the impulse response $\hat{g}(\nu)$ has been estimated, one can estimate the response $\hat{y}$ using

$$\hat{y}(\tau) = \sum_{\nu=0}^{\infty} \hat{g}(\tau-\nu)u(\nu) \tag{49}$$

Then, using (6), one can write

$$\begin{array}{ccc}
\begin{bmatrix} \hat{y}(1) \\ \vdots \\ \\ \\ \\ \\ \hat{y}(\ell) \end{bmatrix} & = & \begin{bmatrix} 0 & 0 & u(0) & 0 & (0)..0 \\ -\hat{y}(1) & 0 & u(1) & u(0) & 0...0 \\ -\hat{y}(2)-y(1) & 0 & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\hat{y}(\ell-1)-\hat{y}(\ell-2) & -\hat{y}(\ell-n) & u(\ell-1) & & u(\ell-n) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \\ \hat{a}_n \\ \hat{b}_1 \\ \\ \hat{b}_n \end{bmatrix} \\
\hat{y} & = & R \qquad\qquad \hat{\beta}
\end{array} \tag{50}$$

One then estimates $(\hat{a}_1,\ldots,\hat{a}_n,\hat{b}_1,\ldots,\hat{b}_n)$ by a least squares computation

$$\hat{\beta} = [R^T R]^{-1} R^T \hat{y} \tag{51}$$

Equations (49)-(51) become especially simple for a step input used for the identification, i.e. $u(0) = 1$, $u(1) = u(2) = u(3) \ldots = 1$. This smooths the noise present in $\hat{g}(\tau)$ because of equation (49).

The algorithm (Stochastic approximation) of eq. (45) converges in the mean-square sense to the true parameter values under the following

Assumptions:

(i) The input $u(k)$ is an independent random variable with $E\{u(k)\} = 0$.

(ii) The noise $n(k)$ is an independent random variable with $E\{n(k)\} = 0$.

(iii) As usual, system order n is assumed.

Sources of error:

It is evident from the assumptions that violation of any of them could cause problems in the estimation scheme - in accuracy or in convergence or in both.

E. Correlation-Cum-Least Squares (Isermann et. at. [1976])

Conceptually this is one of the simplest methods in use. The estimation scheme determines correlation functions and then estimates the parameters of the desired parametric model by the method of least squares. Correlation techniques have been studied extensively and one might mention Buchta [1969], Hayashi [1969], Reid [1969 a&b], Stassen [1969] and Gerdin [1970] as representative of a large body of literature in the area.

Process model:

As in equation (10a)

$$y(k) = -\hat{a}_1 y(k-1) \ \ldots \ -\hat{a}_n y(k-n)+\hat{b}_1 u(k-1)+\ldots+\hat{b}_n u(k-n)+e(k)$$

Assumptions:

Input and output signals are stationary random variables.

The autocorrelation function (acf) of the input then may be written as

Sources of error:

Non-stationarity of input and output signals invalidate the scheme outlined. White noise is not an absolute requirement, it simply facilitates the computation a great deal. ℓ , the number of impulse response values, needs to be chosen.

Chapter V - Concluding Remarks

No discussion of identification techniques would be complete without a comparative evaluation. Surprisingly enough, comparisons of different identification methods are rare in the literature - Isermann et. al [1974], Saridis [1974b], Gustavsson [1972] are recent examples of comparing on-line schemes. With regard to the on-line techniques that have been described in this paper, we could make certain observations. For general linear processes, correlation methods show most advantages compared to the other methods, viz. instrumental variables, stochastic approximation, generalized least squares and least squares (Isermann et. al [1974]).

Very good performance, shortest computation time, 100 percent overall reliability with no problems of poor convergence or instability, choosing only one factor a priori (the number of impulse response values desired) - all these characterize the correlation techniques. As mentioned before, while the method is not restricted to white noise inputs, the computational expense is smallest with such inputs.

Instrumental variable methods perform well for most processes, almost as good as the correlations technique. The principal shorcoming is the choice of a filter factor, which at times could be crucial.

Stochastic approximation methods enjoy the advantages of short computation time and easy implementation. The drawback lies principally in the choice of the gain factors - there is no general rule to use.

Generalized least squares methods perform poorly compared to the above methods, but better than L.S. methods, which, however, take less time than the G.L.S. method.

Several questions remain unanswered. We shall raise only a few of these here. For on-line techniques, can some method be devised to insure better starting estimates? How should gain factors be chosen to improve rate of covergence? How should one deal with the general multivariable problem (the methods discussed were exclusively for single input single output models)? Again, the nature of the disturbances at the output of a process will have a direct effect on the performance of any identification technique. It seems reasonable to address the problem of trying to maintain almost uniform performance against a wide variety of output perturbances. We have already started investigations in this area in the hope of being able to devise some robust procedures for identification.

Bibliography

Albert, A. and Gardner, L. (1967). "Stochastic approximation and non-linear regression", Cambridge, Mass., M.I.T. Press.

Anderson, T.W. (1962). "The choice of the degree of a polynomial regression as a multiple decision problem", Annals Math. Stat., 33, 255-265.

Åström, K.J. (1968). "Lectures on the identification problem - the least squares method", Report, Lund. Inst. Tech. Div. Aut. Control, Sweden.

Åström, K.J. and Bohlin, T. (1966). "Numerical identification of linear dynamic systems from normal operating records", in 'Theory of self adaptive control systems', ed. P.H. Hammond, Plenum Press, New York.

Åström, K.J. and Eykhoff, P. (1971). "System identification- a survey", Automatica, Vol. 7, 123-162.

Åström, K.J. and Söderström, T. (1974). "Uniqueness of the maximum likelihood estimates of the parameters of an ARMA model", IEEE Trans. Aut. Control, Vol. AC-19, No. 6, 769-773.

Balakrishnan, A.V. (1969). "Techniques of system identification", Lecture notes, Institute Electrotecnio, University of Rome.

Baumgartner, S.L. and Rugh, W.J. (1975). "Complete identifications of a class of non-linear systems from steady-state frequency response", IEEE Trans. Circuits & Systems, Vol. CAS-22, No. 9, 753-759.

Box, M.J. and Draper, N.R. (1971). "Factorial Designs, the $|X'X|$ criterion and some related matters", Technometrics, 13, 731-742.

Brockett, R.W. and Mesarovic, M.D. (1965). "The reproducibility of multivariable control systems", JMAA, 11, 548-563.

Buchta, H. (1969). "Method of estimation of random errors in the determination of correlation functions of random infralow frequency signals in control systems", Proc. 4th IFAC Congress, Warsaw, paper 33.4.

Caines, P.E. (1972). "Unique state variable models with applications to the identification of linear systems", Int. J. Control, Vol. 16, 939-944.

Clarke, D.W. (1967). "Generalized-least-squares estimation of the parameters of a dynamic model", IFAC Symp., Identification in Aut. Control Sys., Prague, paper 3.17.

Cumming, I.G. (1970). "Frequency of input signal in identification", 2nd IFAC Sym., Identifications Proc. Para. Est., Prague, papger 7.8.

Denham, M.J. (1974). "Canonical forms for the identification of multivariable linear systems", IEEE Trans. Auto. Control, Vol. AC-19, No. 6, 646-655.

Dickinson, B.W., Kailath, T. and Morf, M. (1974). "Canonical matrix fraction and state-space descriptions for deterministic and stochastic linear systems", IEEE Trans. Auto. Control, Vol. AC-19, No. 6, 656-666.

Dvoretzky, A. (1956). "On stochastic approximation", Proceedings 3rd Berkeley Symp. Math. Stat. and Probability, Vol. 1, 39-55.

Eykhoff, P. (1974). "System identification - parameter and state estimation", John Wiley & Sons, New York.

Eykhoff, P. (1963). "Some fundamental aspects of process-parameter estimation", IEEE Trans. Auto. Control, AC-8, 347-357.

Finigan, B.M. and Rowe, I.H. (1974). "Strongly consistent parameter estimation by the introduction of strong instrumental variables", IEEE Trans. Auto. Control, AC-19, No. 6, 825-830.

Flake, R.H. (1963). "Volterra series representations of nonlinear systems", AIEE Trans. Applic. & Industry, 330-335.

Friedman, B. (1954). "Principles and techniques of applied mathematics, John Wiley, New York.

George, D.A. (1959). "Continuous non-linear systems", M.I.T. Research Lab. Electron, Tech. Report 355.

Gerdin, K. (1970). "Markov modelling via correlation analysis applied to identification of a paper machine in the absence of test signal", 2nd IFAC Symposium Ident. and Process Parameter Estim., Prague, paper 2.8.

Glover, K. and Willems, J.C. (1974). "Parametrizations of linear dynamical systems: canonical forms and identifiability", IEEE Trans. Auto. Control, Vol. AC-19, 640-645.

Godfrey, K.R. (1970). "The application of pseudo-random sequences to industrial processes and nuclear power plant", 2nd IFAC Symp. Indentification and Process Parameter Estimation, Prague, paper 7.1.

Goodwin, G.C., Zarrop, M.B., and Payne, R.L. (1974). "Coupled design of test signals, sampling intervals, and filters for system identification", IEEE Trans. Auto. Control, AC-19, 748-752.

Graupe, D. (1972). "Identification of Systems", Van Nostrand Reinhold, New York.

Gupta, N.K. and Mehra, R.K. (1974). "Computational aspects of maximum likelihood estimation and reduction in sensitivity function calculations", IEEE Trans. Auto. Control, AC-19, 774-783.

Gustavsson, I. (1972). "Comparison of different methods for identification of industrial processes", Automatica, Vol. 8, 127-142.

Harper, T.R. and Rugh, W.J. (1975). "Structural features of factorable Volterra systems", The Johns Hopkins University, EE Report 75-8.

Hastings-James, R. and Sage, M.W. (1969). "Recursive generalized-least-squares procedure for on-line identification of process parameters", Proc. IEE, 116, 2057-2062.

Hayashi, N. (1969). "On a method of correlation analysis for multivariate systems", Proc. 4th IFAC Congress, Warsaw, paper 33.1.

Isermann, R., Baur, U., Bamberger, W., Kneppo, P. and Siebert, H. (1974). "Comparison of six on-line identification and parameter estimation methods", Automatica, Vol. 10, 81-102.

Kalman, R.E. (1963). "Mathematical description of linear dynamical systems", SIAM J. Control, 1, 152-192.

Kalman, R.E. (1973). "Global structure of classes of linear dynamical systems", presented at the NATO ASI Geometric and Algebraic Methods for Non-linear Systems, London, England.

Kashyap, R.L. and Nasburg, R.E. (1974). "Parameter estimation in multivariate stochastic difference equations", IEEE Trans. Auto. Control, AC-19, 784-797.

Keviczky, L. (1975). "'Design of experiments' for the identification of linear dynamic systems", Technometrics, Vol. 17, No. 3, 303-308.

Kiefer, J. (1961). "Optimum design in regression problems, II", Annals Math. Stat., 32, 298-325.

Kiefer, J. and Wolfowitz, J. (1959). "Optimum designs in regression problems", Annals Math. Stat., 30, 271-294.

Mayne, D.Q. (1972 a). "A canonical model for identification of multivariable linear systems", IEEE Trans. Auto. Control, (Tech. notes and Correspondence), AC-17, 728-729.

Mayne, D.Q. (1972 b). "Parametrization and identification of linear multivariable systems", in Lecture Notes in Mathematics, Berlin: Springer, Vol. 294, 56-61.

McGhee, R.B. (1963). "Identification of nonlinear-dynamic systems by regression analysis methods", Ph.D. Dissert., E.E. Dept., Univ. of Sourthern California, Los Angeles, Cal.

Mehra, R.K. (1974). "Optimal input signals for parameter estimation in dynamic systems-survey and new results", IEEE Trans. Auto. Control, AC-19, 753-768.

Mendel, J.M. (1973). "Discrete techniques of parameter estimation: the equation error formulation", New York, Marcel Dekker.

Nieman, R.E., Fisher, D.G. and Seborg, D.E. (1971). "A review of process identification and parameter estimation techniques", Int. J. Control, 13, 209-264.

Nikiforuk, P.N. and Gupta, M.M. (1969). "A bibliography of the properties, generation and control systems applications of shift-register sequences", Int. Journal Control, 9, 217-234.

Pandya, R.N. (1974). "A class of bootstrap estimators and their relationship to the generalized two stage least squares estimators", IEEE Trans. Auto. Control, AC-19, 831-835.

Popov, V.M. (1972). "Invariant description of linear time-invariant controllable systems", SIAM J. Control, Vol. 10, 252-264.

Potts, T.F., Ornstein, G.N. and Clymer, A.B. (1961). "The automatic determination of human and other system parameters", Proc. West. Joint Computer Conf., Los Angeles, 19, 645-660.

Robbins, H. and Monro, S. (1951). "A stochastic approximation method", Ann. Math. Stat., Vol. 22, 400-407.

Reid, K.N. (1969 a). "Lateral dynamics of an idealized moving web", Preprint JACC, 955-963.

Reid, K.N. (1969 b). "Lateral dynamics of an idealized moving web", Preprint JACC, 964-976.

Sage, A.P., and Melsa, J.L. (1971). "System identification", Academic Press, New York.

Sain, M.K., and Massey, J.L. (1969). "Invertibility of linear, time-invariant dynamical systems", IEEE Trans. Auto. Control, AC-14, 141-149.

Saridis, G.N. (1974 a). "Stochastic approximation methods for identification and control - a survey", IEEE Trans. Auto. Control, AC-19, 798-809.

Saridis, G.N. (1974 b). "Comparison of six on-line identification algorithms", Automatica, 10, 69-79.

Schetzen, M. (1974). "A theory of nonlinear system identification", Inter. Journal Control, 20, No. 4, 577-592.

Silverman, L.M. (1969). "Inversion of multivariable linear systems", IEEE Trans. Auto. Control, AC-14, 270-276.

Staley, R.M., and Yue, P.C. (1970). "On system parameter identifiability", Information Sciences, 2, 127-138.

Stassen, H.G. (1969). "The polarity coincidence correlation technique - a useful tool in the analysis of human operator dynamcis", IEEE Trans. Man-Machine Systems, MMS-10, 34-39.

Strobel, H. (1968). "System analysis using deterministic test signals" (in German), V.E.B. Verlag Technik, Berlin.

Tse, E. and Weinert, H.L. (1973). "Structure determination and parameter identification for multivariable stochastic linear systems", presented at the 1973 JACC, Columbus, Ohio.

Tse, E., Weinert, H.L. Anton, J. and Mehra, R.K. (1973). "Model structure determination and identifiability problems in system identification", Annual report (1973), Systems Control Inc., Pal Alto, California.

Vanden Bos, A. (1970). "Estimation of linear system coefficients from noisy responses to binary multifrequency test signals", 2nd IFAC Symp. Identification and process parameter estimation, Prague, paper 7.2.

Vanden Bos, A. (1974). "Estimation of parameters of linear systems using periodic test signals", Report (unnumbered), University of Delft, Netherlands, available from NTIS, Springfield, Va. 22151.

Volterra, V. (1959). "Theory of functionals and of integral and integro-differential equations", Dover, New York.

Weinert, H.L. and Anton, J. (1972). "Canonical forms for multivariable systems identification", presented at the 1972 IEEE Conf. Decision & Control.

Welfonder, E. and Hasenkopf, O. (1970). "Comparison of deterministic and statistical methods for system identification by test measurements at a disturbed industrial control system", 2nd IFAC Symp. Identification & Process Parameter Estimation, Prague, paper 7.5.

Wong, K.Y. and Polak, E. (1967). "Identification of linear discrete-time systems using the instrumental variable method", IEEE Trans. Auto. Control, AC-12, 707-718.

Woodside, C.M. (1970). "Estimation of the order of linear systems", 2nd IFAC Symp. System Identification and Process Parameter Estimation, Prague, paper 17.7.

Wynn, H.P. (1970). "The sequential generation of D-optimum experimental designs", Annals. Math. Stat., 41, 1655-1644.

Young, P.C. (1970). "An instrumental variable method for real-time identification of a noisy process", Automatica, 6, 271-287.

Young, P.C. (1972). "Lectures on Parameter estimation", Summer School: Theory and Practice of Systems Modelling and Identification. Toulouse, France.

Zadeh, L.A. (1962). "From circuit theory to system theory", Proc. IRE, 50, 856-865.

SECTION 2

A BRIEF INTRODUCTION TO TIME SERIES ANALYSIS

Chapter I - Introduction And Scope Of The Report

The problem of system identification, as control engineers call it, is also extensively studied by statisticians under the name of time series analysis. Consider data $\{z_t, t = 1,2,3,...\}$ - in business, economics, engineering and natural sciences - which occur in the form of time series where observations are dependent and where the nature of this dependence is of interest. The techniques available and utilized for the analysis of such series of dependent observations are called time series analysis. That is to say, given observed data, time series analysis is concerned with inference from what was observed to what might have been observed. In general, the aims of time series analysis are

(i) to understand the process generating the time series.

(ii) to predict the behavior of the time series in the future.

Quantitatively, to simulate and predict a time series $\{z_t\}$, and to understand the generating mechanism, one models it as the output of a dynamic system whose input is white noise. Obviously, one may thus consider time series analysis as identification of systems with white noise inputs. Thus, several methods of time series analysis are applicable to identification problems and vice versa. The task is divided into three parts:

(i) Postulating a model (probabilistic) for the process under investigation in which some parameters are unknown.

(ii) Estimating numerical values of parameters for the hypothesized structure.

(iii) Diagnostic checking for the adequacy of the hypothesized model.

The discussion here will be restricted to the principal methods for analyzing stationary time series; that is, a particular version of task (ii). These techniques can be divided into two classes:

a) Frequency domain approach: Mainly, spectral analysis .

b) Time domain approach: Estimating parameters of a hypothesized representation (model).

The subsequent material is based on Parzen [1961, 1967 and 1974], Jenkins [1961] and Box and Jenkins [1970].

With regard to parameter estimation, there are many methods that are used by statisticians. Several of these techniques are familiar to systems engineers- e.g. ordinary and generalized least squares, instrumental variables, maximum likelihood (for a discussion of these and others, see, for instance, Kashyap and Nasburg [1974]). The focus here will be on correlation methods used in time series analysis.

Chapter II - Frequency Domain Approach: Spectral Analysis

A. Introduction

Roughly speaking, spectral analysis is concerned with a study of $\{z_t\}$ from the viewpoint of its frequency content. It has long been traditional among physical scientists to consider a time series as a phenomenon caused by the superposition of sinusoidal waves of various amplitudes, frequencies and phases. The notion of the spectrum is a central one in the analysis of time series. Spectral (harmonic) analysis deals with the theory of decomposition of a time series into sinusoidal components.

One notes that for many time functions $x(t)$, such a decomposition is provided by the Fourier transform

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\omega} x(t) dt \qquad (1)$$

However, no meaning can be attached to this integral for many stochastic processes $\{x(t)\}$ since their sample functions are non-periodic undamped functions and therefore do not belong to the class of functions usually considered by the theories of Fourier series and Fourier integrals. On the other hand, it is reasonable and possible to define a concept of harmonic analysis of stochastic processes - that is, a method of assigning to each frequency ω a measure of its contribution to the 'content' of the process - as was first demonstrated by Weiner [1930] and Khintchine [1933].

A time series may be represented as a superposition of sinusoidal wave forms with 'independent amplitudes' if and only if it is stationary.

A discrete parameter time series $\{z_t, t = 0, \pm 1, \pm 2 \ldots\}$ (or a continuous parameter times series $\{z_t, -\infty < t < \infty\}$) is said to be (wide-sense) stationary if the covariance function

$$R(v) = E[z_t \; z_{t+v}] \tag{2}$$

is a function only of v. Here it is assumed that $E[z_t] = 0$, but all results can be extended to the case where $E[z_t] \neq 0$.

If it is assumed that $\sum_{v=-\infty}^{\infty} |R(v)| < \infty$, then the spectral density

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} e^{-iv\omega} R(v) \qquad -\pi \leq \omega \leq \pi \tag{3}$$

exists and provides a spectral representation

$$R(v) = \int_{-\pi}^{\pi} e^{iv\omega} f(\omega) d\omega \qquad v = 0, \pm 1, \pm 2, \ldots \tag{4}$$

Estimation of f, and hence, the determination of the frequencies ω at which f has local maxima, is the principal concern of spectral analysis.

B. Estimation of Spectral Density $f(\omega)$

Given a time series sample of size T $\{z_t, t = 1, 2, .., T\}$, the sample covariance function $R_T(v)$ and the sample spectral density function (or periodogram) $f_T(\omega)$ are defined by

$$R_T(v) = \frac{1}{T} \sum_{t=1}^{T-|v|} z_t \, z_{t+|v|} \qquad v = 0, \pm 1, \ldots, \pm(T-1)$$
$$= 0 \qquad v = \pm T, \pm(T+1), \ldots \tag{5}$$

$$f_T(\omega) = \frac{1}{2\pi T} \left| \sum_{t=1}^{T} e^{-it\omega} z_t \right|^2 \qquad -\pi \leq \omega \leq \pi \tag{6}$$

It may be shown that $f_T(\omega)$ and $R_T(v)$ are Fourier transforms:

$$f_T(\omega) = \frac{1}{2\pi} \sum_{v=-T}^{T} e^{-iv\omega} R_T(v) \tag{7}$$

$$R_T(v) = \int_{-\pi}^{\pi} e^{iv\omega} f_T(\omega) \, d\omega \qquad v = 0, \pm 1, \ldots \tag{8}$$

However, there <u>is</u> a difficult problem in empirical spectral analysis of a stationary time series possessing a spectral density. This stems from the fact that the obvious estimate of $f(\omega)$, namely $f_T(\omega)$ just defined, is <u>not</u> a consistent estimate of $f(\omega)$ although $R_T(v)$ <u>is</u> a consistent estimate of $R(v)$ at each v.

That is, it may be shown that

$$\lim_{T\to\infty} E[|R_T(v) - R(v)|^2] = 0$$
$$P[\lim_{T\to\infty} R_T(v) = R(v)] = 1 \tag{9}$$

On the other hand, it may be shown that

$$\lim_{T\to\infty} E[e^{iuf_T(\omega)}] = (1-iuf(\omega))^{-1} \tag{10}$$

for every real u and frequency ω.

Consequently, for every real number x,

$$\lim_{T\to\infty} P[f_T(\omega) > x] = e^{-x/f(\omega)} \tag{11}$$

which means that $f_T(\omega)$ is exponentially distributed with mean $f(\omega)$. So, unless $f(\omega) = 0$, there is no mode of probabilistic convergence in which $f_T(\omega) \to f(\omega)$ as $T\to\infty$. However, it is not difficult to construct sequences of estimates of $f(\omega)$ which are consistent.

Since $R_T(v)$ is a consistent estimate of $R(v)$, i.e.

$$\int_{-\infty}^{\infty} e^{iv\omega} f_T(\omega)d\omega \to \int_{-\infty}^{\infty} e^{iv\omega} f(\omega)d\omega \tag{12}$$

it follows (Parzen [1957 a]) that for every bounded continuous function $A(\omega)$,

$$\int_{-\infty}^{\infty} A(\omega)f_T(\omega)d\omega \to \int_{-\infty}^{\infty} A(\omega)f(\omega)d\omega \tag{13}$$

The convergence in (12) and (13) is the same mode as in (9).

A great deal of effort has been expended by researchers in constructing consistent estimates of $f(\omega)$, and Grenander and Rosenblatt [1957] have shown that one need only consider estimates of the form

$$f_T^{\,*}(\omega_0) = \frac{1}{2\pi} \sum_{v=-T}^{T} e^{-iv\omega_0}\, k_T(v)\, R_T(v) \tag{14}$$

where the constants $k_T(v)$ are to be chosen as even functions of v. These estimates may also be written as sample spectral averages

$$f_T^*(\omega_0) = \int_{-\pi}^{\pi} K_T(\omega-\omega_0) f_T(\omega) d\omega \tag{15}$$

where the spectral window

$$K_T(\omega) = \frac{1}{2\pi} \sum_{v=-T}^{T} e^{iv\omega} k_T(\omega) \tag{16}$$

It is assumed that $K_T(\omega)$ achieves its maximum at $\omega = 0$. Its bandwidth may then be written as

$$\beta(K_T) = \frac{\int_{-\pi}^{\pi} K_T(\omega) d\omega}{K_T(0)} \tag{17}$$

The bandwidth of the estimate $f_T^*(\omega)$ is the bandwidth of its spectral window.

In order to specify an estimate $f_T^*(\omega)$, one must state the covariance averaging kernel (also called lag window) $k_T(v)$. There are mainly two methods for generating $k_T(v)$, which include as special cases most of the estimates suggested by various authors.

Let $h(u)$ be a bounded, even, square integrable function, defined for all real u such that

$$|1 - h(u)|/|u|$$

is a bounded function of u.

One class of estimates, called the algebraic type (Parzen [1961]), has $f_T^*(\omega)$ defined by (14) with

$$k_T(v) = h(v/M_T) \tag{18}$$

where the M_T are positive constants tending to 0 as $T \to \infty$ in such a way that $(M_T/T) \to 0$.

The other class of estimates, called the exponential type (Parzen [1961]), has $f_T^*(\omega)$ defined by (14) with

$$k_T(v) = h(A_T e^{\alpha|v|}) \tag{19}$$

where A_T are positive constants $\to 0$ as $T \to \infty$ in such a way that $(\log A_T)/T \to 0$ and α is a positive constant.

The following is a list of different $h(u)$ corresponding to estimates used by various authors.

$$\begin{aligned} h(u) &= 1 - |u| \quad |u| \le 1 \\ &= 0 \qquad\quad\; |u| \ge 1 \end{aligned} \qquad \text{(Bartlett [1950])} \tag{20}$$

$$\begin{aligned} h(u) &= 1/2(1+\cos \pi u) \quad |u| \le 1 \\ &= 0 \qquad\qquad\qquad\;\; |u| \ge 1 \end{aligned} \qquad \text{(Blackman \& Tukey [1959])} \tag{21}$$

$$\begin{aligned} h(u) &= 1 \quad |u| \le 1 \\ &= 0 \quad |u| \ge 1 \end{aligned} \qquad \text{(called truncated periodogram)} \tag{22}$$

$$h(u) = \frac{\sin u}{u} \quad \text{(usually attributed to Daniell) (Parzen [1961])} \tag{23}$$

$$\begin{aligned} h(u) &= 1-|u|^q & |u| \geq 1 & \quad \text{(Parzen [1957b])} \\ &= 0 & \text{otherwise} & \end{aligned} \tag{24}$$

for some constant $q \geq 1$ to be determined

$$\begin{aligned} h(u) &= 1-6u^2+6|u|^3 & |u| \leq 1/2 & \quad \text{(Parzen [1957c])} \\ &= 2(1-|u|)^3 & 1/2 \leq |u| \leq 1 & \\ &= 0 & \text{otherwise} & \end{aligned} \tag{25}$$

Other possible choices for $k_T(v)$ are given in Parzen [1957 d, 1958].

An estimate $f_T^*(\omega)$ is said to be of <u>non-negative</u> type if $f_T^*(\omega) \geq 0 \ \forall\omega$. A necessary and sufficient condition for this is $K_T(\omega) \geq 0 \ \forall\omega$.

Also, assume that the kernel $h(u)$ satisfies

$$|1-h(u)| \leq h_q|u|^q \quad \forall u \tag{26}$$

for some exponent $q > 0$ and constant h_q.

The largest real number q such that kernel $h(u)$ satisfies (26) for some finite h_q is called the <u>characteristic exponent</u>.

An estimate $f_T^*(\omega)$ is said to be of <u>truncated type</u> if there exists a real number $m_T < T$ such that

$$k_T(v) = 0 \quad \text{for} \quad |v| > m_T \tag{27}$$

If there exists a smallest real number m_T satisfying (27), we call it the <u>truncation point</u> of the estimate. The advantages of truncated estimates are reduction in computation since all values of $R_T(v)$ do not have to be computed.

A kernel $h(u)$ satisfying

$$\begin{aligned} h(u) &> 0 \quad \text{for} \quad |u| < 1 \\ h(u) &= 0 \quad \text{for} \quad |u| \geq 1 \end{aligned} \tag{28}$$

gives rise to algebraic estimates with truncation point

$$m_T = M_T \tag{28a}$$

and to exponential estimates with truncation point

$$m_T = -\frac{1}{\alpha} \log A_T \tag{28b}$$

Except for (23), all kernels $h(u)$ listed satisfy (28). m_T, as defined above in (28a) and (28b), is designated as the truncation point of the estimate even if the estimate is not of the truncated type. The statistical properties of estimates are best expressed in terms of m_T.

Parzen [1961] advocates the choice of kernel on the basis of the following:

(i) Bandwidth and variance are inversely proportional for any kernel. The variance of the estimate $f_T^*(\omega)$ for a given bandwidth is lowest for kernels (20), (25) and (21).

(ii) Mean square error criteria indicate the preferability of a kernel with characteristic exponent $q=2$. (23), (25) and (21) meet this requirement.

(iii) Given a truncated $h(u)$ and truncation point m_T, the variance of $f_T^*(\omega)$ is low for (25) and (21). He thus concludes that (21) and (25) are the best competitors for the choice of kernel to use in estimating $f(\omega)$.

Chapter III - Time Domain Approach: Parameter Estimation

A. Introduction

We consider the time series as the output of linear, time invariant, discrete time systems subject to random shocks (white noise). One can describe (parametrize) such a model in several ways; but to use the fewest number of parameters ('parsimonious' parametrization) one employs a mixed autoregressive-moving average (ARMA(p,q)) representation

$$\begin{aligned} z_t+\alpha_1 z_{t-1}+\alpha_2 z_{t-2} +\ldots+ \alpha_p z_{t-p} \\ = u_t+\beta_1 u_{t-1} +\ldots+\beta_q u_{t-q} \end{aligned} \tag{29}$$

or, in operator notation,

$$g(L) z_t = h(L) u_t \tag{30}$$

where L is the lag (or backward shift) operator

$$\begin{aligned} Lz_t &= z_{t-1} \\ g(x) &= 1+\alpha_1 x +\ldots+ \alpha_p x^p \\ h(x) &= 1+\beta_1 x +\ldots+ \beta_q x^q \end{aligned} \tag{31}$$

$\sigma^2 = E[|u_t|^2]$ is the variance of the white noise u_t.

For this model to represent a stationary time series the roots of the characteristic equation $g(x) = 0$ must lie outside the unit circle in the complex plane. An ARMA(p,q) model for a stationary time series has parameters

$\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q,\sigma^2$. (Again, zero mean is assumed; however, that could easily be estimated as another parameter μ). One can see that the ARMA(p,q) model is just the difference equation representation of the familiar single-input-single-output (SISO) state model:

$$x_{k+1} = Ax_k+Bu_k \qquad A = \begin{bmatrix} 0 & I \\ a_1 \ldots\ldots & a_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ \vdots \\ . \\ b_n \end{bmatrix} \tag{31a}$$

$$c = [1\ 0\ 0 \ldots\ldots 0]$$

where $n = p = q$.

The two fundamental processes - autoregressive and moving average schemes - are defined as follows:

Autoregressive (AR(p))

The time series $\{z_t\}$ is assumed to be generated as a linear function of its past values plus a random shock; for some integer p (called the order of the AR scheme), and constants $\alpha_1,\ldots,\alpha_p$,

$$z_t = \alpha_1 z_{t-1} + \ldots\ldots + \alpha_p z_{t-p} + u_t$$

in which the sequence $\{u_t\}$ consists of independent identically distributed (usually assumed normal) random variables.

Moving Average (MA(q))

The sequence $\{z_t\}$ is assumed to be generated as a finite moving (and weighted) average of a sequence of independent, identically distributed random variables $\{u_t\}$; for some integer q, (called the order of the MA scheme),

and constants $\beta_1,\ldots,\beta_q$,

$$z_t = u_t+\beta_1 u_{t-1}+\ldots+\beta_q u_{t-q}$$

As stated earlier, we shall focus our attention on correlation methods for estimating the parameters $\{\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q,\sigma^2,u\}$.

B. Correlation Methods for ARMA Models

Let us consider a (wide-sense) stationary time series $\{z_t, t = 1,\ldots N\}$ with non-zero mean μ and variance σ^2. Since the probability distribution is the same for all times t , one estimates the mean from a set $\{z_t, t = 1,\ldots,N\}$ of observations by

$$\bar{z} = \frac{1}{N}\sum_{t=1}^{N} z_t$$

and the variance σ_z^2 by

$$\hat{\sigma}_z^2 = \frac{1}{N}\sum_{t=1}^{N} (z_t-\bar{z})^2$$

The autocovariance at lag k is defined as

$$\gamma_k = \mathrm{Cov}[z_t, z_{t+k}] = E[(z_t-\mu)(z_{t+k}-\mu)] \qquad (32)$$

The autocorrelation at lag k is

$$\rho_k = \frac{E[(z_t-\mu)(z_{t+k}-\mu)]}{\sqrt{E[(z_t-\mu)^2]E[(z_{t+k}-\mu)^2]}}$$

$$= \frac{E[(z_t-\mu)(z_{t+k}-\mu)]}{\sigma_z^2} \qquad (33)$$

since, for a stationary process, the variance $\sigma_z^2 = \gamma_0$ is the same at time $t+k$ as at time t.

Thus the autocorrelation at lag k is

$$\rho_k = \frac{\gamma_k}{\gamma_0} \tag{34}$$

A number of estimates of the autocorrelation function have been suggested and their properties are discussed in particular by Jenkins and Watts [1968]. It is concluded that the most satisfactory estimate of the k^{th} lag autocorrelation ρ_k is

$$r_k = \frac{c_k}{c_0} \tag{35}$$

where

$$c_k = \frac{1}{N} \sum_{t=1}^{N-k} (z_t - \bar{z})(z_{t+k} - \bar{z}) \qquad k = 0,1,\ldots K \tag{36}$$

is the estimate of autocovariance γ_k, and $\bar{z}$ is the mean of $\{z_t, t = 1,\ldots.,N\}$. K is taken $\geq p+q$.

Following Box and Jenkins [1970], a procedure is now given for estimating the parameters of an ARMA(p,q) model:

$$z_t = \alpha_1 z_{t-1} + \ldots.. + \alpha_p z_{t-p} + \theta_0 + u_t - \beta_1 u_{t-1} - \beta_2 u_{t-2} \ldots.. - \beta_q u_{t-q} \tag{37}$$

p,q are assumed known. $\{u_t\}$ is assumed to be a sequence of independent identically distributed (usually assumed normal) random variables. θ_0 is the overall constant term.

B.1 Mean and variance

$$\bar{z} = \frac{1}{N} \sum_{t=1}^{N} z_t \tag{38}$$

$$s_u^2 = c_0 \qquad \text{where } c_0 \text{ is defined below} \tag{39}$$

B.2 Autocovariance function (acvf)

$$c_k = \frac{1}{N} \sum_{t=1}^{N-k} (z_t - \bar{z})(z_{t+k} - \bar{z}) \qquad \text{where } k = 1, \ldots., K$$

K = no. of acvf values desired $\geq p+q$

(40)

B.3 Autocorrelation function (acf)

$$r_k = \frac{c_k}{c_0} \tag{41}$$

B.4 Estimates (initial) $\hat{\alpha}_0$ of AR parameters

If $p > 0$, solve the set of p linear equations

$$A \underline{\alpha_0} = \underline{x}$$

where

$$A_{ij} = c_{|q+i-j|} \qquad \underline{\alpha_0} = \begin{bmatrix} \alpha_{10} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{p0} \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{bmatrix}$$

$$x_i = c_{q+i} \tag{42}$$

$$i,j = 1,2,\ldots,p$$

B.5 Estimates (initial) $\hat{\beta}_0$ of MA parameters

a. Using the c_k, the following modified c_j' are computed:

$$c'_j = \sum_{i=0}^{p} \sum_{k=0}^{p} \hat{\alpha}_{i0}\hat{\alpha}_{k0}c_{|j+i-k|} \qquad p > 0 \qquad (\hat{\alpha}_{00} = -1)$$

$$= c_j \qquad p = 0$$

$$j = 1,\ldots.,q \tag{43}$$

b. Then the Newton-Raphson algorithm

$$\underline{\tau}^{i+1} = \underline{\tau}^{i} - \underline{h} \tag{44}$$

where $T^i\underline{h} = \underline{f}^i$

is used to calculate the vector τ^{i+1} at the $(i+1)$st iteration from its value τ^i at the i^{th} iteration, where

$$\underline{\tau}^T = (\tau_0,\tau_1,\ldots,\tau_q) \tag{45}$$

$$f_j = \sum_{i=0}^{q-j} \tau_i\tau_{i+j} - c'_j \tag{46}$$

$$\underline{f}^T = (f_0,f_1,\ldots,f_q) \tag{47}$$

$$T = \begin{bmatrix} \tau_0 & \tau_1 & \cdots\cdots & \tau_q \\ \tau_1 & \tau_2 & \cdots\cdots & \tau_q \\ \cdot & & & \\ \cdot & & & 0 \\ \cdot & & & \\ \tau_q & & & \end{bmatrix} + \begin{bmatrix} \tau_0 & \tau_1 & \cdots\cdots & \tau_q \\ & \tau_0 & \tau_1 & \tau_{q-1} \\ & & & \\ & & & \tau_0 \end{bmatrix} \tag{48}$$

with starting values $\tau_0 = \sqrt{c'_0}$, $\tau_1 = \tau_2 = \ldots. = \tau_q = 0$.

c. when $|f_j| < \varepsilon$, $j = 0,1,\ldots,q$ for some pre-specified ε , the process is considered to have converged and the parameter estimates are obtained from the final τ values according to

$$\hat{\beta}_{j0} = -\tau_j/\tau_0 \qquad j = 1,\ldots,q \tag{49}$$

B.6 <u>Estimate $\hat{\theta}_{00}$ of overall constant</u>

$$\hat{\theta}_{00} = \begin{cases} \bar{z}\,(1 - \sum_{i=1}^{p} \hat{\alpha}_{i0}) & p > 0 \\ \bar{z} & p = 0 \end{cases} \tag{50}$$

B.7 <u>Estimate $\hat{\sigma}_u^2$ of white noise variance</u>

$$\hat{\sigma}_u^2 = \begin{cases} \tau_0^2 & q > 0 \\ c_0 - \sum_{i=1}^{p} \hat{\alpha}_{i0} c_i & q = 0 \end{cases} \tag{51}$$

B.8 <u>Backforecasting initial z's</u>

$$E(z_{N-b+\ell}) = \hat{z}_{N-b}(\ell) = \hat{\theta}_{00} + \sum_{i=1}^{p} \hat{\alpha}_{i0}[z_{N-b-i+\ell}] - \sum_{j=1}^{q} \hat{\beta}_{j0}[u_{N-b-j+\ell}] \tag{52}$$

where
$$[z_{N-b-i+\ell}] = \begin{cases} \hat{z}_{N-b}(\ell-i) & \ell > i \\ z_{N-b-i+\ell} & \ell \leq i \end{cases}$$
$$[u_{N-b-j+\ell}] = \begin{cases} 0 & \ell > j \\ z_{N-b-j+\ell} - \hat{z}_{N-b-j+\ell-1}(1) & \ell \leq j \end{cases} \tag{53}$$

with $\ell = 1,2,\ldots L$ (lead time values). The forecasts are obtained for each origin $b = 0,1,\ldots B$ time units before the end of the series. The backforecasts are done up to a negative origin Q' beyond which the difference between z_t and $\hat{\mu}(=\hat{\theta}_{00})$ becomes negligible.

B.9 Calculation of residual sum of squares

a. Having backforecast initial z's , the residuals for a specified set of values of the parameters are computed:

$$\phi_t = (z_t - \hat{\mu}) - \sum_{i=1}^{p} \hat{\alpha}_{i0}(z_{t-i} - \hat{\mu}) + \sum_{j=1}^{q} \hat{\beta}_{j0}\phi_{t-j} \tag{54}$$

$t = Q', Q'+1, \ldots, N$ where Q' is as above in B.8.

b. For given values of the parameters $(\hat{\mu}, \underline{\hat{\alpha}}, \underline{\hat{\beta}})$ the residual sum of squares is computed from

$$S(\mu, \underline{\hat{\alpha}}, \underline{\hat{\beta}}) = \sum_{t=Q'}^{N} \phi_t^2 \tag{55}$$

B.10 Calculation of least squares estimates

The values of the parameters which minimize the residual sum of squares are obtained by a constrained optimization method, proposed by Marquardt [1963] and described in the following form by Box and Jenkins [1970]:

a. Denoting by $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ all the parameters in the model, that is $\underline{\hat{\lambda}} = (\mu, \underline{\hat{\alpha}}, \underline{\hat{\beta}})$, starting values λ_0 are specified along with parameters d and F which constrain the search and a convergence parameter ε. During the search, the values $\phi_t = E[\phi_t | \underline{\hat{\lambda}}, \underline{z}]$ and the derivatives

$$x_{i,t} = -\frac{\partial\phi_t}{\partial\lambda_i} \tag{56}$$

need to be evaluated at each stage of the iterative process.

b. Using the residuals, calculated as described in Section B.9 pt. a (i.e. according to eq. 54), the derivatives are obtained from

$$x_{i,t} = \frac{\{\phi_t(\lambda_{1,0},\ldots,\lambda_{i,0},\ldots,\lambda_{k,0})-\phi_t(\lambda_{1,0},\ldots,\lambda_{i,0}+\delta_i,\ldots,\lambda_{k,0})\}}{\delta_i} \tag{57}$$

c. with $\phi_t, x_{i,t}$ supplied from the current parameter values, the following computations are done:

(i) The k×k matrix

$$A = \{A_{ij}\}$$

where

$$A_{ij} = \sum_{t=Q'}^{N} x_{i,t}\, x_{j,t}$$

(ii) The vector g with elements $g_1, g_2, \ldots, g_k$ where

$$g_i = \sum_{t=Q'}^{N} x_{i,t}\phi_t$$

(iii) The scaling quantities $D_i = \sqrt{A_{ii}}$

d. The modified (scaled and constrained) linearized equations

$$A^* \underline{h}^* = g^* \tag{58}$$

are constructed according to

$$A^*_{ij} = A_{ij}/D_i D_j \quad i \neq j$$

$$A^*_{ii} = 1+d \tag{59}$$

$$g^*_i = g_i/D_i$$

The equations are solved for $\underline{h}^*$, which is scaled back to give the parameter corrections h_j, where

$$h_j = h^*_j/D_j \tag{60}$$

The the new parameter values are constructed from

$$\underline{\lambda} = \underline{\lambda}_0 + \underline{h} \tag{61}$$

and the sum of squares of residuals $S(\underline{\lambda})$ is evaluated.

e. If $S(\underline{\lambda}) < S(\underline{\lambda}_0)$, the corrections $\underline{h}$ are tested. If all are smaller than ε, convergence is assumed. Otherwise, $\underline{\lambda}_0$ is reset to the value $\underline{\lambda}$, d is reduced by a factor F and computation returned to (c).

f. If $S(\underline{\lambda}) > S(\lambda_0)$, the constraint parameter d is reduced by a factor F and computation resumed at (d). In all but exceptional cases, a reduced sum of squares is eventually found. However, an upper bound is placed on d, and if this is exceeded, the search is terminated. When convergence has occurred, either according to the criterion in (e), or it is

assumed to have taken place after a specified number of iterations, the residual variance and the covariance matrix of the estimates are calculated as follows.

B.11 Standard errors and correlation matrix

The estimate of the residual variance is obtained from the value of the sum of the squares function at convergence using

$$\hat{\sigma}_u^2 = \frac{1}{N-p-q-1} \; S(u,\hat{\alpha},\hat{\beta}) \tag{62}$$

and the covariance matrix V of the estimates from

$$V = \{V_{ij}\} = (X^T X)^{-1} \hat{\sigma}_u^2 \tag{63}$$

where X is the regression matrix in the linearized model, calculated at the last iteration in the Marquardt procedure [Eq. (59)].

The standard errors are

$$s_i = \sqrt{V_{ii}} \qquad i = 1,2,\ldots,p+q+1 \tag{64}$$

and the elements R_{ij} of the correlation matrix are obtained from

$$R_{ij} = V_{ij}/\sqrt{V_{ii}\, V_{jj}} \tag{65}$$

Finally, an estimate $\hat{\theta}_0$ of the overall constant term is

$$\hat{\theta}_0 = \hat{\mu}(1 - \sum_{i-1}^{p} \hat{\alpha}_i) \tag{66}$$

C. Correlation Estimation for Noisy Measurement Models

The procedures just described are appropriate for analyzing time series in which the observations z_t are assumed to be uncorrupted by noise. The only perturbances are the random shocks u_t. However, in most cases, the observations themselves are perturbed by additive noise so that one really observes

$$y_t = z_t + n_t \tag{67}$$

where $\{n_t\}$ is a noise sequence independent of the input $\{u_t\}$ and usually assumed white. It would also be helpful if $\{u_t\}$ in general did not have to be white noise. The method to be described considers the more general problem of non-white (or 'colored') noise $\{n_t\}$, the sequence $\{u_t\}$ also being considered as a 'colored' (correlated) sequence of random variables. Of course, considerable simplification occurs if $\{u_t\}$ is a 'white' sequence.

We consider two time series, $\{u_t, t = 1,\ldots,N\}$ and $\{y_t, t = 1,\ldots,N\}$ corresponding to the input and output measurements of a linear process.

Consider the representation of the linear process to be

$$y_t = g_0u_t+g_1u_{t-1}+\ldots+g_mu_{t-m}+n_t \tag{68}$$

where $g_0,g_1,\ldots, g_m$ are impulse response weights for the process. m is the number of impulse response values desired and is at least $= p+q$, where p and q are the orders of

the AR and MA schemes of the ARMA process which one assumes to have generated $\{u_t\}$. Larger values of m increase the model accuracy, so the choice has to be made on the basis of computational cost versus accracy.

First, consider the input sequence $\{u_t\}$ to be the output of an ARMA (p,q) process driven by white noise. That is,

$$d(L)u_t = h(L)a_t \tag{69}$$

where $\{a_t\}$ is a sequence of independent identically distributed random variables.

$$d(x) = 1+\alpha_1 x+\alpha_2 x^2+\ldots+\alpha_p x^p$$

$$h(x) = 1+\beta_1 x+\beta_2 x^2+\ldots+\beta_q x^q$$

Then these $\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q$ are computed by the procedures described previously. Then the following computations are made.

C.1 <u>Differencing and pre-whitening</u>

The input series $\{u_t\}$ and the output series $\{y_t\}$ are differenced to form N values of

$$\begin{aligned} u_t' &= u_t - \bar{u}_t \\ y_t' &= y_t - \bar{y}_t \end{aligned} \tag{70}$$

where $\bar{u}_t$ and $\bar{y}_t$ are the arithmetic means of the u_t and y_t series.

w. : that is to say, we would like to ensure improved perform-

The differenced series are then pre-whitened to give $n' = N-p$ values of the A_t, B_t series according to

$$A_t = u'_t - \sum_{i=1}^{p} \alpha_i u'_{t-i} + \sum_{j=1}^{q} \beta_j A_{t-j}$$

$$B_t = y'_t - \sum_{i=1}^{p} \alpha_i y'_{t-i} + \sum_{j=1}^{q} \beta_j B_{t-j} \qquad (71)$$

$\bar{A}, \bar{B}$ are the means of the A_t, B_t series.

C.2 Pre-whitened output autocorrelation function

$$r_{BB}(k) = \frac{\sum_{j=1}^{n'-k} (B_j-\bar{B})(B_{j+k}-\bar{B})}{\sum_{j=1}^{n'} (B_j-\bar{B})^2} \qquad (72)$$

$$k = 0,1,\ldots,m$$

C.3 Pre-whitened input-output cross-correlation function

$$r_{AB}(k) = \frac{C_{AB}(k)}{S_A S_B} \qquad (73)$$

where

$$C_{AB}(k) = \frac{1}{N} \sum_{j=1}^{n'-k} (A_j-\bar{A})(B_{j+k}-\bar{B}) \qquad k = 0,1,\ldots,m$$

$$C_{AB}(-k) = C_{BA}(k) \qquad k = 1,2,\ldots,m \qquad (74)$$

$$S_A = \sqrt{C_{AA}(0)}$$

$$S_B = \sqrt{C_{BB}(0)} \qquad (75)$$

Thus,

C.4 Impulse response function estimate

$$g_k = \frac{S_B}{S_A} \cdot r_{AB}(k) \qquad k = 0,1,\ldots,m \tag{76}$$

C.5 Noise variance and autocorrelation function

Using the estimates g_k of the impulse response weights, the noise series n_t is regenerated from

$$n_t = y_t - g_0 u_t - g_1 u_{t-1} \cdots - g_n u_{t-n} \qquad \text{where } n \leq m \tag{77}$$

and then the variance and autocorrelation calculated as in Sections B.1, B.2 and B.3. [eqs. 38-41], from the values n_t, $t = 1,2,\ldots,N-n$.

C.6 Parameter estimates

Using the impulse response weights, one may estimate parameters $[a_1,\ldots,a_n,b_1,\ldots,b_n]$ of the SISO state model [eq. 31(a)]. See, for instance, Isermann et al. [1974].

D. Robust Estimation of the Transition Parameter of an AR(1) Model

We conclude this chapter by considering the problem of estimating α in the following AR(1) (or 1st order Markov) model:

$$\begin{aligned} x_k &= \alpha x_{k-1} + u_k \\ z_k &= x_k + v_k \end{aligned} \tag{78}$$

where $\{u_k\}$ and $\{v_k\}$ are independent identically distributed sequences with u and v independent and having distributions

symmetric about the origin. Stationarity is also assumed, which is equivalent to the assumption that $|\alpha|<1$.

The specific model most used is one for which $v_k \equiv 0$ and $\{u_k\}$ are Gaussian. Under such assumptions the estimate for α is the standard least squares one given by

$$\hat{\alpha}_{Ls,n} = \frac{\sum_{i=1}^{n-1} z_i z_{i+1}}{\sum_{i=1}^{n-1} z_i^2} \tag{79}$$

for a set of n observations $z_1,\ldots,z_n$

Unfortunately, $\hat{\alpha}_{Ls,n}$ is asymptotically inefficient when $\{u_k\}$ has a heavy-tailed non-Gaussian distribution so that $\{u_k\}$ has outliers. Also, in the case of non-zero additive effects, i.e. $v_k \neq 0$, the least squares estimate $\hat{\alpha}_{Ls,n}$ is asymptotically biased as it converges to

$$\alpha_0 = \alpha - \alpha . \frac{1}{1+\sigma_x^2/\sigma_v^2} \tag{80}$$

where $\sigma_x^2 \rightarrow$ Variance of x_k

$\sigma_v^2 \rightarrow$ Variance of v_k

This bias can be considerable even when v_k is zero most of the time: say v_k is distributed according to the density

$$f(v) = (1-\varepsilon)\,\delta(v)+\varepsilon G(v|0,\sigma_v^2) \tag{81}$$

with ε small (≤ 0.2)

Thus, v_k is zero $1-\varepsilon$ of the time and comes from a Gaussian distribution ε of the time.

Denby and Martin [1975] and Martin and Jong [1975] have proposed and constructed robust estimates for α in the case of non-Gaussian u and non-zero v by solving the equation

$$\sum_{i=1}^{n-1} g(z_i)\psi(z_{i+1}-\hat{\alpha}_n z_i) = 0 \tag{82}$$

If u and v have finite variances, the estimate α_n' obtained above in (82) converges almost surely to α_b where α_b is the root of the regression equation

$$m(\alpha') = -E\, g(z_i)\psi(z_{i+1}-\alpha' z_i) = 0 \tag{83}$$

and α_b is the same sign as α and $|\alpha_b| \leq |\alpha|$. One can see that (82) gives the least squares estimate $\hat{\alpha}_{Ls,n}$ if $g(\cdot)$ and $\psi(\cdot)$ are taken as the identity functions. As in the least squares case - where $\alpha_b(=\alpha_0)\neq\alpha$ - it is true that, in general, for a particular $g(\cdot)$ and $\psi(\cdot)$, $\alpha_b\neq\alpha$. However, use of suitable $g(\cdot)$ and $\psi(\cdot)$ results in a very small bias when v_k is distributed as in (81), and $0 \leq \varepsilon \leq 0.2$, so that robustness toward bias is obtained as well as robustness toward variance. Martin and Jong [1975] may be consulted for details of asymptotic properties of $\hat{\alpha}_n$ and necessary assumptions and conditions.

Obviously, there remains the need to construct robust estimation procedures for the parameters of general AR(p) and ARMA (p,q models with a similar non-Gaussian assumption on the u_k and $v_k\neq 0$.

Chapter IV - Concluding Remarks

We conclude this discussion by considering the relative merits of frequency domain and time domain approaches to time series analysis - specifically, spectral analysis as compared to correlation analysis (Jenkins [1961]). For a specific application, there are principally two factors to be considered in choosing between correlation and spectral analyses:

(i) The subsequent use of the estimated quantitites.

(ii) The ease of physical interpretation.

These considerations are best illustrated by specific problems:

a) Frequency Response Studies

If the ultimate objective of the analysis is to look at the distribution of variance or power with frequency, then the spectrum provides a direct answer to such problems. Autocorrelations may be used towards such an end, but the computations are far more indirect and complicated. Again, in certain aspects of control system design, the spectrum provides valuable information with regard to the frequency response, such as sharp peaks (resonances). In general, the spectrum has a direct physical interpretation related to the frequency response.

b) Prediction and Simulation

Since prediction and simulation are done in time, it is natural to work in the time domain, as variate values are

required for feeding into the prediction (simulation) model. Spectra are of no use in this type of problem. Also, in several control system design problems, time domain models (such as state space, difference equation) are used extensively. Since parameters are directly estimated as functions of the auto and cross correlations, it is natural to work with these quantities.

c) Exploratory Investigations

Initial studies in many fields are done by building models based on a priori considerations. Both correlation and spectral analysis may be of considerable use in suggesting possible models, depending again upon the type of representation desired.

Because the autocorrelation function and the spectrum are transforms of each other, they are mathematically equivalent. However, as illustrated by the examples just cited, the choice of one or the other will be dictated by considerations of the aims and goals of analyzing a particular time series.

BIBLIOGRAPHY

Bartlett, M.S. (1950). "Periodogram analysis and continuous spectra", Biometrika, Vol. 37, p. 1-16.

Blackman, R.B., and Tukey, J.W. (1959). "The measurement of power spectra from the point of view of communication engineering", New York, Dover.

Box, G.E.P., and Jenkins G.M. (1970). "Time series analysis: forecasting and control", Holden-Day, San Francisco.

Denby, L. and Martin, R.D. (1975). "Robust fitting of auto-regressive models", Annual meeting of ASA, Atlanta, Ga., 1975.

Grenander, U., and Rosenblatt, M. (1957). "Statistical analysis of stationary time series", John Wiley, New York.

Isermann, R., Baur, U., Bamberger, W., Kneppo, P. and Siebert, H. (1974). "Comparison of six on line identification and parameter estimation methods", Automatica, Vol. 10, p. 81-102.

Jenkins, G.M. (1961). "General considerations in the analysis of spectra", Technometrics, 3, p. 133-166.

Jenkins, G.M. and Watts, D.G. (1968). "Spectral Analysis and its applications", Holden-Day, San Francisco.

Kashyap, R.L., and Nasburg, R.E. (1974). "Parameter estimation in multivariate stochastic difference equations", IEEE Trans. Auto. Control, Vol. AC-19, No. 6, p. 784-797.

Khintchine, A. (1933). "Korrelations theorie der stationaren stochastichen Prozesse", Math. Ann., Vol. 109, p. 604-615.

Marquardt, D.W. (1963). "An algorithm for least squares estimation of non-linear parameters", Jour. Soc. Ind. App. Math., 11, p. 431.

Martin, R.D. and Jong, J. (1975). "Assymptotic properties of M-estimates for the transition parameter of a 1st order Markov model", private communication, submitted for publication.

Parzen, E. (1957a). "Conditions that a stochastic process be ergodic", Stanford University Dept. of Statistics Tech. Report 11.

Parzen, E. (1957b), "On consistent estimates of the spectrum of a stationary time series", Ann. Math. Statist., 28, p. 329-348.

Parzen, E. (1957c). Paper presented at the 1957 IMS annual meeting - the kernel was orally suggested by the author. See Parzen [1961].

Parzen, E. (1957d). "On choosing an estimate of the spectral density function of a stationary time series", Ann. Math Statis., Vol. 28, p. 921-932.

Parzen, E. (1958). "On asymptotically efficient consistent estimates of the spectral density function of a stationary time series", J. Roy. Stat. Soc., B, Vol. 20, p. 303-322.

Parzen, E. (1961). "Mathematical considerations in the estimation of spectra", Technometrics, Vol. 3, No. 2, p. 167-190.

Parzen, E. (1967). "Time series analysis papers", Holden-Day, San Francisco

Parzen, E. (1974). "Some recent advances in time series modeling", Trans. Auto. Contr. (IEEE), Vol. AC-19, No. 6, p. 723-729.

Weiner, N. (1930). "Generalized harmonic analysis", Acta. Math., Vol. 55, p. 117.

SECTION 3

ROBUST ESTIMATION OF CROSS CORRELATIONS FOR PARAMETER IDENTIFICATION OF SINGLE-INPUT, SINGLE-OUTPUT, LINEAR, TIME INVARIANT, NOISY SYSTEMS MODELED IN DISCRETE TIME

PRECEDING PAGE BLANK-NOT FILMED

Chapter 1 Problem Statement and Proposed Solution

A. Introduction and Formulation of the Problem

We recall (from Sections 1&2) that the on-line identification of single input single output linear, time invariant, noisy systems modeled in discrete time is of considerable general interest, and, accordingly, has been studied most extensively. However, in the construction of identification schemes, there remain interesting questions which have yet to receive adequate consideration - specifically, for one, the modeling of the non-measurable output noise. It is obvious that the nature of this corrupting noise has a direct effect on the performance of any identification algorithm. In general, though, there have been somewhat extreme approaches to modeling the measurement noise. At one end of the scale, the output noise is assumed to be known perfectly. For example, the maximum likelihood method assumes Gaussian noise with known parameters. On the other hand, there are methods which assume (or require) little knowledge of the measurement noise - correlation analysis, for example, only requires stationarity of the input sequence and the output noise, and that they be mututally orthogonal or uncorrelated. Since one can be expected, in reality, to have some, if not complete, knowledge of the operating environment, it seems reasonable to utilize a model in which the output noise is neither completely specified (as a Gaussian process, for example), nor left completely unspecified. We thus formulate

the following identification problem.

Consider the on-line identification of single input single output linear, time invariant, noisy systems modeled in discrete time

$$x_k = Ax_{k-1} + bu_{k-1}$$

with observations (1)

$$z_k = cx_k + w_k$$

where

$$A = \begin{bmatrix} -a_1 & & \\ \vdots & & I \\ -a_n & & 0\ 0 - - - 0 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad c = [\ 1\ 0\ 0\ 0\ -\ -\ -\ ]$$

under the assumptions that

(i) the order of the system, n, is known;

(ii) the system is completely controllable and observable, and stable;

(iii) the distribution of the measurement noise w is given by the mixture $F(w) = (1-\varepsilon)\ K(w) + \varepsilon\ C(w)$, (1a)

where $K(\cdot)$ is known (i.e. completely specified) and C belongs to some broad class of distributions. This is to say, w_k comes $(1-\varepsilon)$ of the time from a known distribution and ε of the

time from a distribution $C \varepsilon \not{C}$. For the rest of the report, we use the following equivalent representation of (1): (see, for example, Luders & Narendra [1973])

$$\begin{aligned} y_k &= -a_1 y_{k-1} - a_2 y_{k-2} - a_3 y_{k-3} \cdots - a_n y_{k-n} \\ &\quad + b_1 u_{k-1} + b_2 u_{k-2} + \cdots + b_n u_{k-n} \\ z_k &= y_k + w_k \end{aligned} \tag{2}$$

where $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are the unknown parameters to be identified.

Some investigations of this problem, in somewhat different formulations, have been initiated by Nasburg and Kashyap [1975], Denby and Martin [1975] and Martin and Jong [1975].

B. Review of Cross-correlations method - possible shortcomings

Referring again to Section 1, one recalls that the method of cross correlations seems to be most advantageous compared to other methods (Isermann et al [1974], Saridis [1974]) for general linear processes. Here, we briefly review the essence of the method. We have, under the assumption of stationarity, that the autocorrelation of the input sequence $\{u_k\}$ is

$$\begin{aligned} R_{uu}(m) &= E[u_k u_{k-m}] \\ &= \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} u_i u_{i-m} \end{aligned} \tag{3}$$

and the cross correlation of the inputs $\{u_k\}$ and outputs

$\{z_k\}$ is

$$R_{zu}(m) = E\ [z_k\ u_{k-m}] \qquad (4)$$

$$= \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} z_i\ u_{i-m}$$

now,

$$\begin{aligned} R_{zu}(m) &= E\ [z_k\ u_{k-m}] \\ &= E\ [(y_k + w_k)\ u_{k-m}] \qquad (5) \\ &= R_{yu}(m) + R_{wu}(m) \end{aligned}$$

Since $\{u_k\}$and $\{w_k\}$ are mutually uncorrelated,

$$R_{wu}(m) = 0.$$

Thus, $R_{zu}(m) = R_{yu}(m)$ (6)

The convolution equation

$$R_{yu}(m) = \sum_{i=0}^{\infty} g(i)\ R_{uu}\ (m-i) \qquad (7)$$

relates R_{yu} to R_{uu} through $g(i)$, the impulse response of the system. Thus, if an input sequence $\{u_k\}$ of known autocorrelations is used, then estimates of $g(i)$ can be computed from estimates of $R_{yu}(= \hat{R}_{zu})$ and R_{uu}. In particular, if $\{u_k\}$ is chosen as discrete white noise, we have

$$R_{uu}\ (i) = R_{uu}\ (0).\ \delta_i \quad \text{where } \delta_i = 1 \quad i = 0$$
$$= 0 \quad i \neq 0$$

w_k; that is to say, we would like to ensure improved performance (over the scheme L) in terms of lower error and lower standard deviations of the error and the estimates in the cases where the contamination level ε is, say, 5-20% and the contaminating distribution $C(\cdot)$ has a large variance and/or has heavy tails (i.e. more outliers). We will not mind if the price to be paid in adopting such a scheme N is that it is worse - but not by 'much' - than L in the case where $\varepsilon = 0$, i.e. w_k comes from a totally known distribution. We will call the scheme N robust in the manner of Huber [1972] and Anscombe [1960]: 'I am willing to pay a premium (a loss of efficiency of, say, 5 to 10% at the ideal model) to safeguard against ill effects caused by small deviations from it; although I am happy if the procedure performs well also under large deviations, I do not really care - inferences based upon a grossly wrong statistical model may have little physical significance.'

C. Proposal for a Robust estimate of R_{zu}

We propose to use a non-linear scheme N for estimating the correlations R_{zu}. It should be mentioned that non-linear estimates of autocorrelations have been proposed and used by Huzii [1963, 1970] for stationary Gaussian processes and by Rodemich [1966] for more general stationary processes. However, the motivation of both these investigators was the simplification of the computational effort and not the desensitization of the estimation technique to different noise processes.

Thus,

$$\hat{g}(m) = \frac{\hat{R}_{yu}(m)}{R_{uu}(0)} = \frac{\hat{R}_{zu}(m)}{R_{uu}(0)} \tag{8}$$

The cross correlation estimate, $\hat{R}_{zu}(m)$, may be computed recursively (on-line) by

$$\hat{R}_{zu}(m,k) = \hat{R}_{zu}(m,k-1) + \frac{1}{k+1} [z_k u_{k-m} - \hat{R}_{zu}(m,k-1)] \tag{9}$$

Henceforth, we shall refer to (9) as scheme L (for linear). Using the impulse response estimates $\hat{g}(m)$, one can then estimate the parameters $a_1,\ldots,a_n,b_1,\ldots,b_n$ of (2) by the method outlined on p.32 of Section 1. Our principal concern in this investigation is with regard to the computation of the cross correlations $\hat{R}_{zu}$.

The possible shortcoming of scheme L is imbedded in the generality of the noise w_k. While it is only required that $\{w_k\}$ and $\{u_k\}$ be stationary and mutually uncorrelated, it is natural to expect that the performance of the algorithm L in terms of rate of convergence, error of the estimation scheme and standard deviation of the error, bias of the estimates and standard deviations of the estimates - is dependent on the distribution of $\{w_k\}$. We propose to modify L (the scheme for the on-line computation of $\hat{R}_{zu}$) for the case where w_k, in addition to satisfying the usual assumptions for the correlations method, comes from a distribution F $(\cdot)$ as specified in the problem formulation (1a). The goal is to make the computational algorithm N less sensitive to the distribution of

Consider again scheme L and the recursive algorithm used:

$$\hat{R}_{zu}(m,k) = \hat{R}_{zu}(m,k-1) + \frac{1}{k+1}\left[z_k u_{k-m} - \hat{R}_{zu}(m,k-1)\right]$$

Thus, as each new observation z_k is measured, the estimate of $R_{zu}(m)$ for (k-1) measurements, $\hat{R}_{zu}(m,k-1)$, is updated by adding $\frac{1}{k+1}\left[z_k u_{k-m} - \hat{R}_{zu}(m,k-1)\right]$.

We will call the term in parentheses the innovation I_k for the k^{th} measurement z_k. Our proposal is to use $H(I_k)$ instead of I_k in the recursive equation for R_{zu}, where $H(\cdot)$ is an odd, non-linear function as defined below (also see fig. 1). Thus, we have

$$\hat{R}_{zu}(m,k) = \hat{R}_{zu}(m,k-1) + \frac{1}{k+1} H\left[z_k u_{k-m} - \hat{R}_{zu}(m,k-1)\right] \tag{10}$$

$$\begin{aligned} H(x) &= s_1 x & |x| &\le d_1 \\ &= s_1 d_1 \operatorname{sgn}(x) & d_1 \le |x| &\le d_2 \\ &= s_2\left(x - \left(\frac{s_2 d_2 - s_1 d_1}{s_2}\right)\operatorname{sgn}(x)\right) & d_2 \le |x| &\le \left(\frac{s_2 d_2 - s_1 d_1}{s_2}\right) \\ &= 0 & |x| &\ge \left(\frac{s_2 d_2 - s_1 d_1}{s_2}\right) \end{aligned} \tag{11}$$

$$s_1 > 0,\ s_2 \le 0,\ d_2 > d_1 > 0,\ d_1 < \infty$$

$H(\cdot)$ is the estimator proposed by Hampel [1968,1971] to be used in the robust estimation of the location parameter of a distribution. It can be seen that $H(\cdot)$ is the identity function (i.e. the linear case) for $s_1 = s_2 = 1$, $d_1 = d_2 = \infty$. If we choose $s_2 = 0$ and $d_2 = \infty$, H is then the well known limiter function used by Huber [1964] for robust estimation and

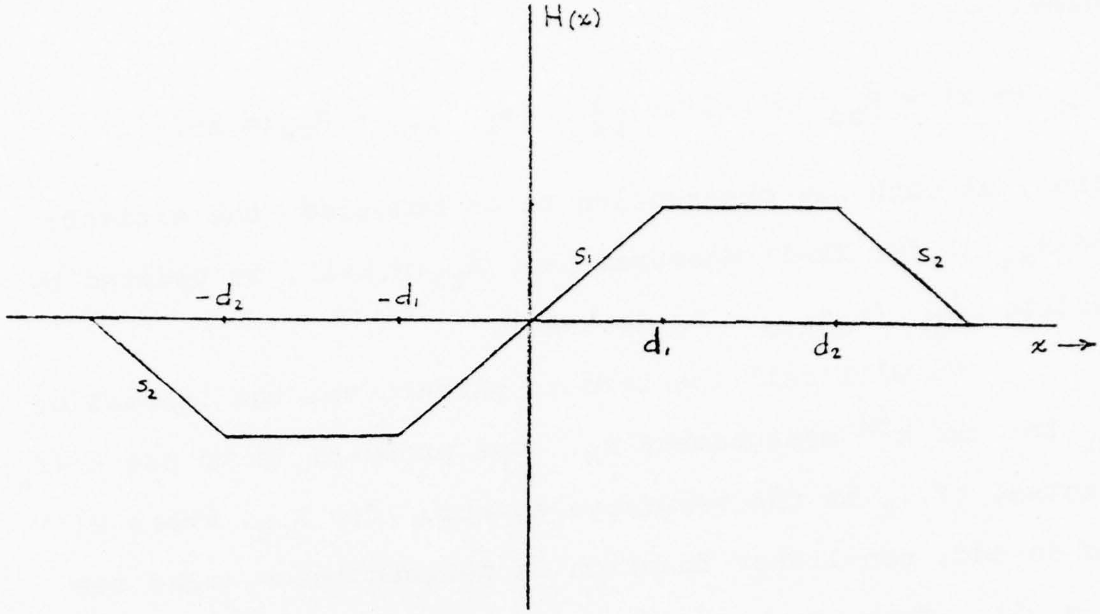


Figure 1

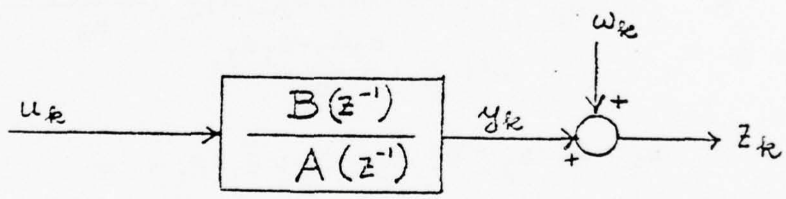


Figure 2

it has a constant value $s_1 d_1$ sgn(x) for $|x| \geq d_1$.

The motivation for using a non linear H comes from an examination of the innovations $\{I_k\}$. Although the effect of I_k is damped at each successive stage (only $I_k/(k+1)$ is added to $\hat{R}_{zu}(m,k-1)$), the characteristics of the noise in the $\{z_k\}$ obviously influence the performance of the algorithm. We feel that the proposed H, using different values for the parameters s_1, s_2, d_1, d_2 - henceforth called the Hampel parameters - will tend to stabilize the recursive computation of $\hat{R}_{zu}$ against a variety of perturbances $\{w_k\}$. The recursive algorithm (9) used for the linear scheme L has been claimed by Saridis [1974], without proof, to be a stochastic approximation algorithm (Robbins-Monro [1951], Dvoretzky [1956]), which would ensure convergence of the scheme L to the true values $R_{zu}(m)$ and give well-defined asymptotic properties of the estimates. We have not been able to show, at the present time, that the scheme N of egn (10) using H(·) as defined in (11) satisfies the conditions of stochastic approximation (SA) (as in Dvoretzky [1956], Sacks [1958]). On the other hand, we are optimistic of obtaining these results since the linear scheme L satisfies the SA conditions (Saridis [1974]) and the non-linear function H used in scheme N is bounded for every non-linear estimator that we choose. This follows on recalling from (11), with the stated restrictions on the Hampel parameters, that

$$H(x) \leq s_1 d_1 \operatorname{sgn}(x) \quad \forall x$$

To examine the performance of different non-linear estimators for R_{zu} as compared to the linear estimator, we have carried out an extensive digital computer simulation for the identification of three linear processes which have been used as test cases by Isermann et al [1974]. We have used various distributions for $\{w_k\}$ to examine, in particular, the effect of the measurement noise on the identification schemes. The details of the simulation and summarized results are reported in the ensuing chapter.

Chapter II Simulation of process identification using different estimators for R_{zu}

A. Details of Simulation

The following three simulated linear processes were used (Fig. 2).

(i) Second order oscillating process

$$\frac{Y_k}{u_k} = G_{21}(z^{-1}) = \frac{B(z^{-1})}{A(z^{-1})} = \frac{b_1 z^{-1} + b_2 z^{-2}}{1+a_1 z^{-1}+a_2 z^{-2}} \tag{12}$$

where $z^{-1} \rightarrow$ the unit delayor

$a_1 = -1.500$ $a_2 = 0.700$

$b_1 = 1.000$ $b_2 = 0.500$

(ii) Third order low pass process

$$G_3(z^{-1}) = \frac{b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} \tag{13}$$

$a_1 = -1.500$ $a_2 = 0.705$ $a_3 = -0.100$

$b_1 = 0.065$ $b_2 = 0.048$ $b_3 = -0.008$

(iii) Second order non minimum phase process

$$G_{22}(z^{-1}) = \frac{B(z^{-1})}{A(z^{-1})} = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{14}$$

$a_1 = -1.425$ $a_2 = 0.496$

$b_1 = -0.102$ $b_2 = 0.173$

Process (i) was used by Åström and Bohlin [1966], Hastings-James and Sage [1969], Gustavsson[1972], Gentil [1972] and Isermann et al [1974]. Process (ii) has the same parameters as the third order process used by Isermann et al [1974], with the difference that we used no time delay. Process (iii) was proposed and used by Isermann et al [1974].

The output y_k of the process to be identified was corrupted by w_k, for which the following 20 noise distributions were used. Distributions are specified by means and standard deviations; G and L refer to Gaussian and Laplacian distributions respectively. We have chosen our mixture family F to be the contaminated Gaussian class, where the contaminating distribution C in synmetric, since this is the most important case from a practical viewpoint. The contaminations are either Gaussian with larger variances or Laplacian (double exponential) of different variances.

(Please see Table 1 on next page.)

TABLE 1

Noise #	Nominal (Known) Distribution	Contaminating Distribution	Contamination level ε
1	G(0,1)	L(0,1)	.05
2			.10
3			.15
4		L(0,3.16)	.05
5			.10
6			.15
7		L(0,10)	.05
8			.10
9			.03
10		G(0,3.16)	.05
11			.10
12			.15
13		G(0,10)	.05
14			.10
15			.03
*** 16	G(0,1)	-	-
17	G(0,3.16)	-	-
18	G(0,10)	-	-
* 19	G(0,1)	L(0,3.16)	.20
* 20		G(0,3.16)	.20

*** Used as basis for comparing estimators for different noises

* Only used for the second order processes

These distributions are all part of a library of random functions developed by the author and A. H. El-Sawy in the Electrical Engineering Department of The Johns Hopkins University.

The input $\{u_k\}$ was discrete binary white noise generated by using

$$
\begin{aligned}
v &= \text{ran}\ (\emptyset)\ -0.5 \\
u &= \text{sign}\ (0.5,\ v)
\end{aligned}
\tag{15}
$$

The call ran (Ø) returns an uniformly distributed number r, $0 \leq r \leq 1$, and is reproducible.

The sign (x,y) function returns $|x| \cdot$ (Sign of y). Thus, $\{u_k\}$ was a white binary sequence of amplitude +0.5 and -0.5. So, the autocorrelation $R_{uu}(0) = (0.5)^2 = 0.25$.

The following estimators H were used in the on-line computation of $\hat{R}_{zu}(m)$:

TABLE 2

#	Estimator type		Hampel Parameters		
		s_1	s_2	d_1	d_2
1	Linear	1	1	∞	∞
2	Limiter 1 (Huber [1964])	1	0	2σ	∞
3	Limiter 2 "	1	0	1.5σ	∞
4	Limiter 3 "	1	0	σ	∞
5	Limiter 4 "	1	0	0.7σ	∞
6	Hampel 12A (Hampel [1968]	1	-0.27	1.2σ	3.5σ
7	Hampel 25A (")	1	-0.50	2.5σ	4.5σ

Note: σ is the standard deviation of the known distribution.

The identification algorithms for each estimator were run for 3000 stages of the process, and parameter estimates were computed every 300 stages using the scheme in Section 1, p. 32. The number of impulse response values, ℓ, was taken as 2n. It should be noted that to improve the estimates (i.e. to reduce error and standard deviations of the estimates) ℓ has to be chosen very much larger than 2n. Otherwise, because of dependence of $g(\ell)$ for $\ell > 2n$ on $g(1),\ldots,g(2n)$, noisy measurements increase the variance of estimates and introduce large bias if ℓ is not much larger than 2n.

AD-A038 294 BALLISTIC RESEARCH LABS ABERDEEN PROVING GROUND MD F/G 20/1

ROBUST IDENTIFICATION OF LINEAR SYSTEMS.(U)

FEB 77 V D VANDELINDE

UNCLASSIFIED BRL-1960 NL

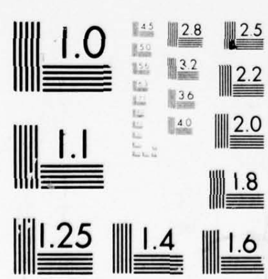
2 OF 3

AD A038294

SIFIED

2 OF 3

AD
A038294



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

TABLE 5

The error was then computed as the mean squared parameter errors related to mean squared true parameters:

$$\text{Error} = \frac{\| \Delta \theta_i \|}{\| \theta_i \|} = \left[\frac{\sum_{i=1}^{2n} (\Delta \theta_i)^2}{\sum_{i=1}^{2n} \theta_i^2} \right]^{1/2} \tag{16}$$

where $(\theta_1,\ldots,\theta_{2n}) = (a_1,\ldots,a_n,b_1,\ldots,b_n)$ are the true parameter values and $(\hat{\theta}_1,\ldots,\hat{\theta}_{2n}) = (\hat{a}_1,\ldots,\hat{a}_n,\hat{b}_1,\ldots,\hat{b}_n)$ are the parameter estimates, and $\Delta\theta_i = \theta_i - \hat{\theta}_i$.

To obtain better comparisons of the different identification schemes, 30 Monte Carlo runs of each identification run (of 3000 stages) were carried out. Then the means and standard deviations of the error (as defined above) and the parameter estimates were computed. If j refers to each Monte Carlo run of the identification algorithm,

$$\text{Mean error} = \bar{e} = \frac{1}{30} \sum_{j=1}^{30} \text{error } (j) \tag{17}$$

$$\text{std. dev. of error} = \text{sd}(e) = \left[\frac{1}{30} \sum_{j=1}^{30} (\text{error } (j) - \bar{e})^2 \right]^{1/2}$$

mean parameter estimates are

$$\bar{\theta}_i = 1/30 \sum_{j=1}^{30} \theta_i \ (j) \qquad i=1,\ldots,2n$$

std. dev. of parameter estimates are (18)

$$sd(\theta_i) = \left[1/30 \sum_{j=1}^{30} (\theta_i(j) - \bar{\theta}_i)^2 \right]^{1/2}$$

All the simulations were carried out using the facilities of the Westinghouse Computer Laboratory of the Electrical Engineering Department of The Johns Hopkins University.

The computer was a PDP 11/45.

B. Summarized results

We have summarized the data for all the 406 examples [140 each for the 2nd order processes, 126 for the 3rd order process] in the following manner:

For each process, we have chosen as our basis the values (after 3000 stages) of

(i) the error

(ii) the standard deviations of the error

(iii) the std. deviations of the parameter estimates

for the LINEAR estimator when w_k is distributed according to noise 16, i.e. G(0,1). We have then taken the corresponding values for all estimators for all the noise distributions, and computed them as percentages of the base values [i.e. the linear estimator values for $w_k \sim G(0,1)$] for that process. If the percentage exceeded 999, we took it, for the purposes of summary, as 999.9.Tables 3-7 are the summarized data for process (i), tables 8-12 are the summarized data for process (ii) and tables 13-17 are the summarized data for process (iii). To use the tables, one needs to refer also to tables 1 and 2. Any row in the tables 3-17 has its entries arranged as follows:

The first entry (any number from 1 to 20) specifies the distribution for w_k (as per table 1). The next entry (any number from 1 to 7) indicates the particular estimator used (as per table 2). The next two entries are, for the just specified noise distribution and estimator, the error and

standard deviation of the error as percentages of the corresponding values for the linear estimator with $w_k \sim G(0,1)$ (noise 16). The remaining entries (6 for the 3^{rd} order case and 4 for the 2^{nd} order cases) are the percentages relative to the linear estimator for noise 16 ($G(0,1)$) of the standard deviations of the parameter estimates.

B.1. Summarized results for process (i)

The 2σ limiter has the most uniform performance over all the distributions for w_k in terms of the least variation in the 6 comparative values. For a given noise distribution, there is not much to choose between the best estimators in terms of bias. The 2σ limiter has lower values than the linear estimator for all but three noise distributions, and is very close to the best performance for all distributions. In most cases, it is the best estimator. Even in the very noisy case, i.e. $G(0,10)$, where almost all entries in table 7 (for noise 18) exceed the upper limit (999), the non linear estimates work much better than the linear estimator. The robustness of the non linear estimates is well observed for the cases of high contamination (ε=0.15, .20) and/or high contaminating variances (σ_c^2 = 10 or 100). One can see that almost all the non linear estimators have better performance than the linear estimator in extreme cases like 7, i.e. when $F(w) = .95\ G(0,1) + .05\ L(0,10)$.

Next to the 2σ limiter, the Hampel 25A estimator seems to have the most uniform behavior over different noise distributions.

It should be noted that in almost all cases, the parameter estimates have little bias. The non-linear schemes do not significantly reduce the bias for the linear estimator. Finally, an important point of note is that most of the non-linear estimators show marked improvement over the linear estimator in the cases when C, the contaminating distribution, has more outliers. In our simulations, this was achieved by choosing C as different Laplacian distributions.

It is evident that, for process (i), our proposed scheme N successfully robustizes, in the sense defined before, the on-line identification of process parameters by using non-linear estimators for R_{zu}.

B.2. Summarized results for process (ii)

This process, a third order low pass system, was found to be difficult to identify and produced poor estimates with any estimation scheme, linear or non-linear. Very large bias of estimates, high standard deviations of error and of the parameter estimates were a common feature of every simulation run. The summarized data presented in table 8-12 can easily lead to confusion. They show, for example, that the worst performance of the linear estimator is for $w \sim G(0,1)$ and the best is for $w \sim .97\ G(0,1) + .03\ L(0,10)$! Thus, the performance of all estimators for different distributions of w are relative to a very poor basis and this fact should temper any use of tables 8-12.

There are only two points we would like to note with respect to the simulations for process (ii). Firstly, Isermann et al [1974] also report poor parameter identification

for process (ii) using correlation analysis. Secondly, an examination of tables 8-12 (with the aforementioned caution) shows, interestingly enough, that several of the non-linear estimators perform better than the linear estimator for a large number of distributions. The Hampel 12A, the 2σ limiter and the 1.5σ limiter seem to have the most uniform performance over all the distributions.

B.2. Summarized results for process (iii).

It is difficult to determine which is the most robust estimator for this process. The limiter at σ, the limiter at 0.7σ and the Hampel 12A - all show good performance over the different noise distributions. All of these estimators give considerably better results than the linear estimator for all but two of the distributions. The 1.5σ limiter, 2σ limiter and the Hampel 25A estimator also have improved performance over the linear estimator, but not as uniformly as the first three. Of the latter group, the 1.5σ limiter works best, followed by the 2σ limiter and then the Hampel 25A estimator. The non-linear estimators for this process also reduce the bias of the estimates significantly in several instances.

Finally, the improvement over the linear estimator is again most marked when the contaminations in w come from the heavy-tailed Laplacian distribution. As may be expected, the degree of improvement achieved by using the non-linear schemes increases with increasing level of contamination ε.

Thus, for process (iii), our proposed scheme N provides a robust procedure for the on-line identification of parameters through non-linear estimation of R_{zu}. The additional benefit which arises for this process from the use of these non-linear estimators is a significant reduction of the bias in the parameter estimates for several noise distributions.

We have included, after Table 17, the computer printout for 5 noise distributions, 2 each for the second order processes and 1 for the third order process. For each noise, identification results for all the estimators are given.

Chapter III Concluding Remarks

It is clear from the results of the simulation that our proposal provides a robust procedure for estimating cross correlations to be used for parameter identification. We would like here to clarify an issue which is fundamental to the problem.

The parameter identification problem of Chapter I can be split into two parts. The first part, which involves the estimation of $g(m)$ from estimates of $R_{zu}(m)$, is non-parametric and essentially infinite-dimensional. The second part is the transformation from this infinite-dimensional characterization to the finite dimensional parametric representation of (2). We have not addressed ourselves at all to the problems associated with the second part. This was done by following the same procedure as in Isermann et al [1974], viz. a least squares estimation of the parameters $a_1,\ldots,a_n,b_1,\ldots,b_n$ which uses values of the uncorrupted system output y computed from $\hat{g}(1),\ldots,\hat{g}(2n)$ by the Wiener-Hopf equation.

Errors could easily arise from the use of such a procedure, especially since least squares estimation requires uncorrelated residuals to give unbiased estimates.

It would thus be interesting to investigate the following question: given a particular process to be identified, consider only the estimation of the impulse response $g(m)$ using our proposed scheme N for computing $\hat{R}_{zu}$. Isermann [1974] reported that the correlation analysis produces the

best estimates of impulse response in terms of bias and variance of the estimates. We feel it is interesting to examine the effect of our scheme N on the estimation of g(m).

We mention in closing that though our scheme N seems to have worked well, an immediate question is whether there would have been additional improvement if we had used, like Isermann [1976], 18-22 values of the impulse response instead of 2n(i.e. 4 or 6). We chose 2n values for reasons of computational cost, but hope to investigate the above question (i.e., using ℓ=18-22) in the near future.

Bibliography

Anscombe, F. (1960). "Rejection of outliers", Technometrics, 2, 123-147.

Åström, K. J. and Bohlin, T. (1966). "Numerical identification of linear dynamic systems from normal operating records", in ' Theory of self adaptive control systems',ed. P. H. Hammond, Plenum Press, New York.

Denby, L. and Martin, R. D. (1975). "Robust fitting of auto-regressive models", Annual meeting of ASA, Atlanta, Ga., 1975.

Dvoretzky, A. (1956). "On Stochastic Approximation", 3rd Berkeley Symposium on Math. Statistics and Probability, Vol. 1.

Gentil, S. (1972). "Etude comparitive de diverses méthodes statistiques d'identification de Systèmes Dynamiques", Thesis, Scientific University of Grenoble.

Gustavsson, I. (1972). "Comparison of different methods for identification of industrial processes", Automatica, Vol. 8, 127-142.

Hampel, F. R. (1968). "Contributions to the theory of robust estimation", Ph.D dissertation, Univ. of California, Berkeley.

Hampel, F. R. (1971). "A general qualitative definition of robustness", Annals Math. Stat., 42, 1887-1896.

Hastings-James, R. and Sage, M. W. (1969). "Recursive generalized-least-squares procedure for on-line identification of process parameters", Proc. IEE, 116, 2057-2062.

Huber, P. J. (1964). "Robust estimation of a location parameter", Annals Math. Stat., 35, 72-100.

Huber, P. J. (1972). "The 1972 Wald Lecture: Robust statistics: a review", Annals Math. Stat., 43, 1041-1067.

Huzii, M. (1963). "On a simplified method of the estimation of the correlogram for a stationary Gaussian process", (Tokyo) Inst. of Stat. Math. Annals, 14, 259-268.

Huzii, M. (1970). "On the variance of a simplified estimate of a correlogram", (Tokyo) Inst. of Stat. Math. Annals, 22, 527-534.

Isermann, R., Baur, U., Bamberger, W., Kneppo, P. and Siebert, H. (1974). "Comparison of six on-line identification and parameter estimation methods", Automatica, Vol. 10, 81-102.

Lüders, G. and Narendra, K. S. (1973). "An adaptive observer and identifier for a linear system", IEEE Trans. Auto. Control, AC-18, 496-499.

Martin, R. D. and Jong, J. (1975). "Asymptotic properties of M-estimates for the transition parameter of a 1st order Markov model", private communication, submitted for publication.

Nasburg, R. E. and Kashyap, R. L. (1975). "Robust parameter estimation in dynamic systems", Proc. JHU Conference on Information Sciences and Systems.

Robbin, H. and Monro, S. (1951). "A stochastic approximation method", Annals Math. Stat., 22, 400-407.

Rodemich, E. R. (1966). "Spectral estimates using nonlinear functions", Annals Math. Stat., 37, 1237-1256.

Sacks, J. (1958). "Asymptotic distribution of stochastic approximation procedures", Annals Math. Stat., 29, 3730405.

Saridis, G. N. (1974). "Comparison of six on-line identificarion algorithms", Automatica, 10, 69-79.

TABLE 3

SUMMARY OF DATA FOR FIRST SECOND ORDER EXAMPLE

The base values are (for the linear estimator for G(0,1))

							F(w)
	.1151	.0592	.116	.158	.081	.143	
1 1	107.3	128.4	106.0	108.9	102.5	122.4	
1 2	105.0	126.2	102.6	105.7	102.5	120.3	
1 3	101.3	136.3	105.2	103.2	103.7	123.1	.95 G(0,1) + .05 L(0,1)
1 4	123.8	147.3	117.2	118.4	119.8	152.4	
1 5	189.4	274.2	200.0	183.5	156.8	249.0	
1 6	110.1	138.9	115.5	105.7	109.9	132.9	
1 7	106.3	128.9	105.2	107.0	103.7	122.4	
2 1	93.9	91.2	87.1	91.1	91.4	102.8	
2 2	95.8	91.9	90.5	91.1	96.3	104.2	
2 3	100.1	81.8	91.4	91.8	102.5	105.6	.9 G(0,1) + .1 L(0,1)
2 4	116.0	93.2	109.5	107.0	112.3	119.6	
2 5	167.1	152.2	161.2	152.5	144.4	186.0	
2 6	105.9	83.6	90.5	100.6	107.4	109.8	
2 7	95.1	93.1	87.1	93.0	92.6	103.5	
3 1	96.3	149.8	105.2	110.8	91.4	118.9	
3 2	101.8	138.9	110.3	107.6	93.8	119.6	
3 3	111.9	125.8	112.1	112.0	101.2	125.9	.85 G(0,1) + .15 L(0,1)
3 4	144.0	159.8	141.4	149.4	123.5	157.3	
3 5	229.0	266.4	238.6	234.2	172.8	251.7	
3 6	124.6	131.8	132.8	120.3	112.3	134.3	
3 7	98.4	147.0	110.3	108.9	91.4	118.2	
4 1	124.2	163.5	134.5	127.8	103.7	146.9	
4 2	117.7	147.0	119.0	121.5	102.5	137.1	
4 3	117.9	151.0	126.7	119.0	103.7	139.9	
4 4	141.0	164.0	141.4	134.2	118.5	170.6	.95 G(0,1) + .05 L(0,3.16)
4 5	213.9	319.6	227.6	213.3	160.5	281.8	
4 6	126.5	153.4	126.7	126.6	109.9	151.0	
4 7	117.2	150.8	124.1	120.3	102.5	137.1	

TABLE 4

								F(w)
5	1	116.6	120.3	119.0	117.1	[illegible]	124.5	
5	2	106.7	100.5	101.7	109.5	[illegible]	107.0	
5	3	113.1	101.7	105.2	114.6	109.9	113.3	
5	4	138.6	139.0	135.3	143.7	124.7	142.0	.9 G(0,1) + .1 L(0,3.16)
5	5	196.4	231.6	201.7	200.6	159.3	226.6	
5	6	119.7	119.4	117.2	124.1	117.3	119.6	
5	7	103.5	104.1	100.0	108.2	96.3	104.2	
6	1	155.8	231.3	175.9	170.9	98.8	189.5	
6	2	131.0	163.9	137.9	140.5	97.5	147.6	
6	3	140.3	147.1	149.1	140.5	103.7	151.7	
6	4	180.1	208.1	183.6	191.8	127.2	201.4	.85 G(0,1) + .15 L(0,3.16)
6	5	283.7	415.5	339.7	296.8	174.1	355.2	
6	6	156.0	146.5	154.3	158.2	113.6	164.3	
6	7	130.8	169.4	141.4	140.5	97.5	148.3	
7	1	257.9	342.1	277.6	284.8	137.0	295.1	
7	2	134.7	172.1	139.7	143.7	102.5	155.9	
7	3	130.2	162.7	133.6	136.7	102.5	151.7	
7	4	153.0	172.3	155.2	145.6	124.7	180.4	.95 G(0,1) + .05 L(0,10)
7	5	231.8	375.2	258.6	238.6	170.4	311.2	
7	6	135.9	150.2	140.5	131.6	108.6	154.5	
7	7	144.9	168.8	145.7	148.7	103.7	163.6	
8	1	301.7	598.8	353.4	386.1	142.0	435.0	
8	2	133.4	162.7	140.5	148.7	112.3	137.8	
8	3	141.3	170.6	141.4	158.2	122.2	151.0	
8	4	183.9	203.7	181.9	202.5	137.0	193.0	
8	5	245.1	326.4	269.0	271.5	169.1	278.3	.9 G(0,1) + .1 L(0,10)
8	6	169.9	182.6	163.8	178.5	133.3	183.9	
8	7	145.4	171.1	142.2	158.2	108.6	153.8	

TABLE 5

9	1	-212.9	297.8	244.0	241.1	119.8	236.4	F(w)
9	2	114.1	116.2	112.9	110.8	100.0	114.0	
9	3	115.2	122.6	107.8	115.2	104.9	115.4	
9	4	148.7	212.0	150.9	158.2	124.7	165.0	.97 G(0,1) + .03 L(0,10)
9	5	256.1	420.1	251.7	283.5	169.1	318.9	
9	6	121.6	152.2	120.7	121.5	113.6	128.0	
9	7	108.8	104.4	102.6	103.8	98.8	109.8	
10	1	102.2	138.7	97.4	101.3	96.3	127.3	
10	2	99.9	141.7	99.1	98.7	98.8	125.9	
10	3	109.8	157.6	106.0	105.7	108.6	142.0	
10	4	133.0	188.3	130.2	113.9	137.0	162.9	.95 G(0,1) + .05 G(0,3.16)
10	5	191.7	245.6	162.9	145.6	181.5	214.0	
10	6	121.1	185.1	111.2	119.0	123.5	158.0	
10	7	100.6	142.6	97.4	102.5	97.5	126.6	
11	1	106.6	148.5	104.3	105.1	98.8	138.5	
11	2	105.9	151.2	108.6	103.2	101.2	135.7	
11	3	116.2	163.0	115.5	108.9	108.6	151.7	.9 G(0,1) + .1 G(0,3.16)
11	4	141.9	202.2	131.0	127.8	137.0	180.4	
11	5	198.6	254.2	168.1	157.0	179.0	228.0	
11	6	130.1	192.1	118.1	126.6	123.5	171.3	
11	7	103.5	151.4	112.1	100.0	98.8	133.6	
12	1	133.0	209.3	140.5	143.7	108.6	176.9	
12	2	124.3	189.2	128.4	136.7	106.2	159.4	
12	3	133.8	191.7	146.6	136.1	114.8	167.8	
12	4	156.4	225.7	153.4	155.1	142.0	194.4	
12	5	214.2	252.2	194.8	174.1	181.5	236.4	.85 G(0,1) + .15 G(0,3.16)
12	6	146.8	214.7	145.7	152.5	128.4	186.0	
12	7	123.7	191.6	144.8	130.4	102.5	158.0	

TABLE 6

								F(w)
13	1	188.4	235.6	131.9	186.1	142.0	221.0	
13	2	110.6	142.1	107.8	101.9	107.4	140.6	
13	3	115.5	159.5	117.2	103.8	116.0	151.0	
13	4	137.4	206.1	130.2	122.8	148.1	179.0	.95 G(0,1) + .05 G(0,10)
13	5	192.7	308.4	194.0	164.6	190.1	237.1	
13	6	134.9	174.5	142.2	125.3	125.9	162.9	
13	7	113.5	141.2	112.1	108.2	106.2	142.0	
14	1	214.5	309.5	238.8	216.5	163.0	272.7	
14	2	121.5	210.6	128.4	136.7	114.8	170.6	
14	3	127.4	204.4	139.7	127.8	119.8	176.2	
14	4	154.8	234.3	150.0	148.7	148.1	205.6	.9 G(0,1) + .1 G(0,10)
14	5	211.5	330.2	227.6	188.0	191.4	262.9	
14	6	158.3	212.8	165.5	157.6	139.5	194.4	
14	7	130.2	183.3	142.2	138.0	114.8	158.7	
15	1	148.0	233.6	164.7	161.4	108.6	187.4	
15	2	98.3	147.3	96.6	100.6	100.0	125.2	
15	3	106.3	152.9	107.8	101.9	108.6	135.0	.97 G(0,1) + .03 G(0,10)
15	4	132.3	191.2	125.0	118.4	142.0	162.2	
15	5	191.9	253.7	158.6	153.8	182.7	218.9	
15	6	117.0	183.1	124.1	117.1	127.2	145.5	
15	7	98.8	154.7	106.0	104.4	101.2	124.5	
16	1	100.0	100.0	100.0	100.0	100.0	100.0	
16	2	97.7	103.4	100.0	100.6	101.2	95.8	
16	3	106.0	118.6	103.4	110.8	107.4	111.2	
16	4	146.5	176.4	141.4	146.8	127.2	169.2	
16	5	226.2	338.3	228.4	241.8	169.1	273.4	G(0,1)
16	6	124.4	138.7	123.3	122.2	114.8	137.8	
16	7	98.4	99.2	100.0	99.4	100.0	96.5	

TABLE 7

17	1	303.4	380.2	307.8	325.3	184.0	331.5	F(w)
17	2	297.0	351.5	318.1	306.3	182.7	314.7	
17	3	293.7	343.2	317.2	301.3	181.5	307.7	
17	4	290.4	314.5	286.2	303.8	185.2	291.6	
17	5	318.7	377.4	330.2	339.2	204.9	323.8	G(0, 3.16)
17	6	292.2	331.8	308.6	298.7	182.7	301.4	
17	7	299.7	361.5	323.3	310.8	182.7	318.9	
18	1	999.9	999.9	999.9	999.9	561.7	999.9	
18	2	999.9	999.9	999.9	999.9	502.5	999.9	
18	3	999.9	999.9	999.9	999.9	533.3	999.9	
18	4	999.9	999.9	999.9	999.9	501.2	999.9	G(0,10)
18	5	999.9	999.9	999.9	999.9	600.0	999.9	
18	6	999.9	999.9	999.9	999.9	519.8	999.9	
18	7	999.9	999.9	999.9	999.9	561.7	999.9	
19	1	188.4	265.5	203.4	191.1	104.9	249.7	
19	2	149.8	223.6	172.4	153.8	92.6	200.0	
19	3	134.0	213.0	149.1	143.0	91.4	186.0	
19	4	143.1	268.6	175.9	157.0	108.6	211.9	
19	5	274.0	686.3	375.0	386.7	165.4	424.5	.8 G(0,1) + .2 L(0,3.16)
19	6	126.1	215.0	145.7	134.8	98.8	180.4	
19	7	154.1	219.9	169.8	157.0	96.3	203.5	
20	1	150.0	223.5	149.1	158.2	121.0	199.3	
20	2	141.2	196.8	154.3	139.2	112.3	179.0	
20	3	152.2	199.3	151.7	149.4	124.7	190.9	
20	4	174.3	220.6	181.9	155.7	149.4	204.2	.8 G(0,1) + .2 G(0,3.16)
20	5	248.7	335.0	226.7	216.5	197.5	286.0	
20	6	167.5	221.8	150.9	171.5	136.3	205.6	
20	7	141.1	219.6	150.9	151.3	112.3	186.7	

TABLE 8

SUMMARY OF DATA FOR THIRD ORDER EXAMPLE

The base values are (for the linear estimator for G(0,1)

									F(w)
	5.71	14.1	5.65	3.58	1.89	14.5	13.5	11.6	
1 1	33.9	9.9	13.4	14.7	15.6	12.3	12.3	12.4	
1 2	32.5	12.5	12.1	12.1	12.0	13.5	13.5	13.5	
1 3	40.8	15.5	19.0	20.5	22.2	18.7	18.7	18.7	
1 4	48.8	34.8	42.0	46.4	48.9	36.9	37.0	37.1	.95 G(0,1)+.05 L(0,1)
1 5	34.8	14.0	16.8	17.7	18.3	18.1	18.1	18.1	
1 6	38.9	25.3	23.3	22.9	23.4	28.6	28.6	28.4	
1 7	37.5	10.9	14.2	15.5	16.4	13.5	13.5	13.5	
2 1	86.0	89.0	54.1	50.2	47.3	90.6	90.3	89.6	
2 2	49.7	24.0	28.0	30.5	32.9	28.4	28.4	28.4	
2 3	82.6	93.7	80.3	86.3	87.3	97.5	97.4	97.1	
2 4	468.3	964.5	970.9	999.9	999.9	937.3	938.3	938.3	.9 G(0,1)+.1 L(0,1)
2 5	29.7	5.2	8.2	8.6	8.9	8.8	8.7	8.7	
2 6	31.6	8.7	13.1	13.7	13.9	13.0	13.0	13.0	
2 7	73.5	89.0	54.1	50.1	47.3	90.6	90.3	89.6	
3 1	35.2	12.5	15.1	15.5	15.6	17.3	17.3	17.3	
3 2	45.8	17.7	19.3	20.5	21.5	18.8	18.8	18.8	
3 3	31.8	10.6	13.5	14.1	14.5	14.8	14.8	14.8	.85 G(0,1)+.15 L(0,1)
3 4	44.0	17.2	21.9	23.2	24.4	23.1	23.1	23.1	
3 5	60.7	64.7	62.4	64.2	65.7	65.9	65.9	65.8	
3 6	49.8	25.5	28.7	30.4	32.0	30.8	30.8	30.8	
3 7	37.5	19.4	21.4	22.3	23.8	23.2	23.2	23.1	
4 1	41.9	28.3	26.7	26.4	25.4	30.8	30.8	30.7	
4 2	35.1	16.8	22.9	25.5	27.6	19.8	19.9	20.0	
4 3	32.2	10.2	11.8	11.6	11.1	15.0	15.0	14.9	
4 4	59.7	51.9	43.5	42.6	42.2	56.2	56.1	55.8	.95 G(0,1)+.05 L(0,3.16)
4 5	31.9	10.6	11.9	11.8	11.5	15.0	15.0	14.9	
4 6	56.4	34.6	43.5	48.1	51.8	38.9	39.0	39.0	
4 7	74.0	109.0	96.8	102.2	102.4	109.8	109.6	109.6	

TABLE 9

										F(w)
5	1	33.5	9.8	12.6	13.0	13.0	14.3	14.3	14.3	
5	2	27.9	9.1	12.9	14.1	14.7	11.8	11.8	11.8	
5	3	182.7	324.0	216.1	214.6	210.3	327.4	326.4	324.2	.9 G(0,1)+.1 L(0,3.16)
5	4	29.3	7.8	11.2	12.0	12.6	11.9	11.9	11.9	
5	5	72.8	49.8	41.6	42.4	41.9	57.1	56.9	56.6	
5	6	41.0	17.5	20.0	21.0	21.9	22.0	22.0	22.0	
5	7	691.3	999.9	943.4	825.4	752.1	999.9	999.9	999.9	
6	1	38.5	23.7	27.7	29.8	32.6	26.6	26.6	26.7	
6	2	36.3	16.7	18.2	20.2	20.6	18.7	18.6	18.6	
6	3	32.4	13.7	13.1	13.3	13.5	17.7	17.6	17.5	
6	4	162.3	289.1	305.4	325.1	364.1	281.2	281.6	282.0	.85 G(0,1)+.15 L(0,3.1
6	5	132.2	192.3	140.3	150.3	154.5	192.6	192.2	191.1	
6	6	68.4	96.5	65.0	70.4	67.3	99.3	99.0	98.3	
6	7	35.8	19.2	15.5	15.2	15.0	23.1	23.0	22.9	
7	1	49.9	18.9	19.0	18.5	18.3	26.7	26.6	26.4	
7	2	34.7	11.9	15.0	15.7	16.1	16.7	16.7	16.6	
7	3	26.0	4.1	8.1	8.8	9.8	7.9	7.9	7.9	
7	4	25.5	8.7	10.3	10.9	11.6	10.2	10.2	10.2	
7	5	42.9	19.0	18.3	18.7	18.7	24.7	24.6	24.5	.95 G(0,1)+.05 L(0,10)
7	6	38.0	12.0	16.0	17.3	18.8	17.4	17.4	17.4	
7	7	53.6	29.0	32.7	34.8	37.8	32.7	32.7	32.7	
8	1	62.0	56.5	36.0	35.8	31.8	61.7	61.4	60.9	
8	2	253.8	473.4	540.0	598.3	640.0	453.4	455.0	457.1	
8	3	24.3	3.3	6.0	6.4	7.0	6.1	6.0	6.0	.9 G(0,1)+.1 L(0,10)
8	4	72.6	95.3	69.5	69.6	73.1	97.6	97.3	96.7	
8	5	168.3	239.9	231.0	250.3	267.6	239.5	239.1	239.4	
8	6	32.9	11.5	15.9	17.0	17.7	15.1	15.1	15.2	
8	7	26.7	5.8	7.9	8.4	8.8	8.9	8.9	8.8	

TABLE 10

									F(w)
9 1	27.1	4.3	6.8	7.6	8.3	5.8	5.8	5.8	
9 2	40.7	16.7	19.0	20.1	20.4	22.0	22.0	21.9	
9 3	31.9	12.3	14.8	15.7	15.9	15.8	15.8	15.8	
9 4	46.1	41.3	34.4	36.3	39.5	42.5	42.5	42.3	
9 5	30.0	10.2	12.9	13.2	13.9	13.1	13.0	13.0	.97 G(0,1)+.03 L(0,10)
9 6	27.4	7.2	9.7	10.6	11.8	9.9	9.9	9.9	
9 7	65.5	73.9	51.1	48.7	42.9	77.9	77.6	77.1	
10 1	31.0	8.6	8.9	9.2	9.5	9.8	9.8	9.8	
10 2	59.2	41.8	36.8	37.3	35.2	43.5	43.4	43.3	
10 3	46.2	45.1	41.7	46.9	46.3	46.8	46.8	46.8	
10 4	41.3	24.2	28.6	30.8	32.2	25.8	25.8	25.9	
10 5	65.0	44.7	38.5	39.7	39.6	50.2	50.1	49.8	.95 G(0,1)+.05 G(0,3.16
10 6	44.7	32.5	22.9	22.8	22.0	35.5	35.4	35.1	
10 7	70.2	54.3	45.5	50.3	51.6	57.1	57.0	56.8	
11 1	28.4	5.2	9.0	9.4	9.7	9.4	9.4	9.4	
11 2	31.0	12.1	14.5	16.2	18.7	13.3	13.4	13.4	
11 3	41.7	22.2	20.7	21.6	22.3	25.7	25.7	25.6	
11 4	64.4	46.6	44.0	46.5	47.5	52.0	52.0	51.8	
11 5	63.9	56.8	58.9	65.1	73.8	58.6	58.7	58.7	.9 G(0,1)+.1 G(0,3.16)
11 6	41.7	27.3	26.0	28.4	30.7	30.9	30.8	30.7	
11 7	71.3	93.4	89.8	94.8	96.9	92.1	92.2	92.1	
12 1	33.2	11.4	12.0	13.2	15.3	13.2	13.2	13.1	
12 2	32.9	14.0	16.5	17.5	18.1	17.5	17.5	17.5	
12 3	26.4	5.7	6.6	7.0	7.2	7.4	7.4	7.4	
12 4	540.7	999.9	881.8	902.2	878.5	999.9	999.9	999.9	.85 G(0,1)+.15 G(0,3.16)
12 5	44.3	22.3	23.8	24.6	24.7	27.2	27.2	27.1	
12 6	41.7	36.9	32.8	32.0	28.2	39.6	39.5	39.4	
12 7	29.2	7.0	11.0	11.6	11.7	11.3	11.3	11.3	

TABLE 11

									F(w)
13	1	59.3	47.5	45.6	59.8	77.5	50.8	50.9	50.8	
13	2	31.3	9.1	13.9	14.9	15.6	12.9	12.9	12.9	
13	3	30.8	7.1	7.5	7.8	8.2	8.6	8.6	8.6	
13	4	199.5	170.8	119.1	129.9	127.0	174.3	173.8	172.7	
13	5	69.2	63.9	72.4	79.5	82.8	66.9	67.0	67.1	.95 G(0,1)+.05 G(0,10)
13	6	46.4	46.5	31.2	33.7	29.9	47.8	47.6	47.3	
13	7	36.1	13.6	13.0	12.7	12.0	17.5	17.4	17.3	
14	1	43.1	32.9	22.7	27.7	9.6	37.1	36.4	36.5	
14	2	29.3	5.2	9.8	10.5	11.3	9.8	9.8	9.8	
14	3	36.5	21.7	25.1	27.0	29.4	22.9	23.0	23.0	
14	4	31.1	7.3	9.5	10.4	11.4	9.9	9.9	9.9	.9 G(0,1)+.1 G(0,10)
14	5	31.5	7.9	12.1	13.2	14.3	10.9	10.9	10.9	
14	6	94.0	141.4	95.7	101.5	103.4	144.7	144.3	143.4	
14		28.0	5.7	8.5	9.0	9.3	8.8	8.8	8.8	
15	1	46.3	30.3	29.5	30.3	31.3	33.9	33.9	33.7	
15	2	32.4	12.6	16.3	17.2	18.2	15.6	15.6	15.6	
15	3	26.3	5.0	7.8	8.2	8.4	8.0	8.0	8.0	.97 G(0,1)+.03 G(0,10)
15	4	31.0	11.5	12.2	13.0	14.0	15.0	15.0	14.9	
15	6	35.0	11.8	16.4	17.8	19.1	15.1	15.2	15.2	
15	5	74.9	86.3	84.7	87.4	89.0	85.1	85.1	85.1	
15	7	26.9	5.7	8.1	8.6	8.9	9.6	9.5	9.5	
16	1	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	
16	2	83.9	79.8	81.4	80.9	79.1	83.9	83.9	83.8	
16	3	48.9	42.5	41.1	42.5	43.3	43.2	43.2	43.1	G(0,1)
16	4	34.4	9.6	13.7	14.7	15.8	13.9	13.9	13.9	
16	5	39.2	24.5	29.7	32.9	36.8	27.0	27.0	27.1	
16	6	67.2	53.8	50.3	51.9	52.2	57.7	57.6	57.4	
16	7	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	

TABLE 12

										F(w)
17	1	33.3	11.6	11.9	9.3	13.6	16.0	15.7	15.1	
17	2	31.6	11.8	11.0	7.7	12.4	15.7	15.4	14.8	
17	3	53.7	27.3	36.3	40.3	44.6	32.5	32.5	32.4	G(0,3.16)
17	4	87.6	113.4	111.2	105.9	96.2	114.4	114.3	113.9	
17	5	30.1	6.5	9.7	10.6	12.4	11.5	11.4	11.1	
17	6	63.4	46.7	46.9	57.5	55.9	51.6	51.8	51.2	
17	7	33.3	11.6	11.9	9.3	13.6	16.0	15.7	15.1	
18	1	55.3	63.1	76.2	106.8	120.9	45.4	66.5	75.6	
18	2	55.6	63.1	76.0	106.6	120.6	45.4	66.5	75.6	
18	3	34.1	11.8	23.1	24.5	45.2	15.9	15.9	14.6	
18	4	52.6	46.0	94.3	98.7	56.9	44.9	41.8	30.6	G(0,10)
18	5	51.2	43.3	78.0	49.0	172.8	39.5	35.5	47.0	
18	6	36.8	13.9	21.1	24.7	41.3	18.4	17.6	17.5	
18	7	55.8	63.1	76.2	106.8	120.9	45.4	66.5	75.6	

TABLE 13

SUMMARY OF DATA FOR SECOND SECOND ORDER EXAMPLE

The base values are (for the linear estimator for G(0,1)

	.5221	.3437	.545	.593	.063	.098	F(w)
1 1	102.5	102.2	97.4	131.0	95.2	91.8	
1 2	104.4	119.7	119.4	130.5	106.3	96.9	
1 3	100.9	113.2	107.7	128.8	95.2	78.6	.95 G(0,1)+.05 L(0,1)
1 4	101.3	100.7	119.1	114.5	95.2	79.6	
1 6	100.6	108.5	110.3	123.3	114.3	101.0	
1 7	104.4	101.1	102.0	131.0	93.7	91.8	
1 5	95.6	104.8	108.3	115.7	95.2	89.8	
2 1	90.9	94.1	78.7	126.5	79.4	78.6	
2 2	87.1	66.4	73.8	120.6	79.4	75.5	
2 3	69.6	102.7	103.1	114.8	82.5	65.3	
2 4	93.5	117.1	115.8	120.6	100.0	64.7	
2 5	104.6	163.8	160.2	130.5	261.9	133.7	.9 G(0,1)+.1 L(0,1)
2 6	85.6	93.9	90.1	112.3	79.4	66.3	
2 7	88.5	91.0	73.0	124.1	84.1	77.6	
3 1	123.0	156.6	181.7	128.7	150.8	168.4	
3 2	126.6	187.8	207.7	139.0	66.7	115.3	
3 3	123.6	151.3	191.7	117.2	96.8	101.0	
3 4	117.3	139.5	156.5	134.7	66.7	107.1	
3 5	175.3	542.8	495.0	272.0	93.7	240.8	
3 6	109.5	123.0	138.5	123.1	101.6	118.4	.85 G(0,1)+.15 L(0,1)
3 7	179.3	430.0	493.4	133.2	303.2	275.5	
4 1	115.2	154.8	137.1	157.3	114.3	99.0	
4 2	117.9	179.0	125.1	151.1	723.8	356.1	
4 3	103.2	102.1	104.2	125.5	119.0	84.7	
4 4	96.0	104.4	97.4	119.7	114.3	79.6	
4 5	93.3	110.9	113.4	111.1	92.1	80.6	.95 G(0,1)+.05 L(0,3.16)
4 6	99.3	104.0	97.2	124.8	101.6	80.6	
4 7	101.7	111.0	106.4	124.1	192.1	95.9	

TABLE 14

								F(w)
5	1	135.6	144.4	162.0	154.0	112.7	143.9	
5	2	100.3	102.6	81.7	135.4	103.2	110.2	
5	3	97.4	108.6	98.0	127.5	100.0	92.9	
5	4	96.7	94.3	105.0	109.9	85.7	89.8	.9 G(0,1)+.1 L(0,3.16)
5	5	113.5	175.8	168.4	143.7	103.2	125.5	
5	6	99.8	121.8	103.7	133.9	111.1	74.5	
5	7	100.1	81.4	94.7	112.5	100.0	106.1	
6	1	190.6	235.9	248.6	219.7	88.9	167.3	
6	2	138.3	160.8	171.2	161.7	123.8	151.0	
6	3	117.0	106.3	122.2	135.6	79.4	99.0	
6	4	122.1	116.1	137.2	132.4	82.5	105.1	
6	5	124.4	113.6	135.2	137.4	82.5	118.4	
6	6	120.3	139.2	0.0	999.9	999.9	45.9	.85 G(0,1) +.15 L(0,3.16)
6	7	138.9	154.3	170.6	145.0	141.3	173.5	
7	1	225.2	362.5	345.7	282.8	230.2	519.4	
7	2	108.6	114.8	108.1	134.6	103.2	101.0	
7	3	101.6	98.0	104.4	118.0	127.0	90.8	
7	4	100.1	111.6	111.4	123.1	165.1	122.4	
7	5	96.1	129.1	129.5	117.5	109.5	84.7	
7	6	96.6	87.7	81.1	116.0	122.2	75.5	.95 G(0,1)+.05 L(0,10)
7	7	110.7	100.4	104.0	129.7	95.2	112.2	
8	1	252.2	285.5	256.1	285.2	184.1	225.5	
8	2	126.6	168.5	131.7	166.3	100.0	137.8	
8	3	139.4	277.3	213.4	221.2	157.1	193.9	
8	4	178.2	728.8	710.8	109.8	104.8	365.3	
8	5	177.9	652.8	508.3	403.2	95.2	289.8	.9 G(0,1)+.1 L(0,10)
8	6	110.2	119.1	120.9	127.5	79.4	107.1	
8	7	149.5	210.7	188.4	186.7	90.5	141.8	

TABLE 15

								F(w)
9	1	200.3	240.8	255.3	203.7	282.5	357.1	
9	2	100.2	71.3	101.8	106.7	115.9	104.1	
9	3	86.2	67.7	88.6	94.1	95.2	101.0	
9	4	65.6	59.1	89.5	87.0	95.2	87.8	.97 G(0,1)+.03 L(0,10)
9	5	90.7	58.9	91.6	94.1	87.3	88.8	
9	6	88.3	60.2	91.7	90.9	90.5	104.1	
9	7	102.9	67.2	99.4	110.3	77.8	111.2	
10	1	108.2	164.7	90.6	175.4	136.5	98.0	
10	2	98.8	139.9	87.5	149.2	134.9	106.1	
10	3	99.2	139.6	90.8	150.9	101.6	101.0	
10	4	125.4	322.5	150.6	292.4	184.1	164.3	.95 G(0,1)+.05 G(0,3.16)
10	5	92.0	97.9	81.3	115.2	147.6	96.9	
10	6	100.4	125.6	97.8	140.0	128.6	101.0	
10	7	98.1	122.1	82.4	136.6	155.6	103.1	
11	1	104.3	109.2	91.0	126.3	114.3	124.5	
11	2	122.2	235.2	104.4	235.4	133.3	132.7	
11	3	108.9	194.1	92.1	198.5	81.0	123.5	
11	4	143.5	392.9	175.2	355.8	130.2	134.7	.9 G(0,1)+.1 G(0,3.16)
11	5	98.3	90.3	88.3	114.7	141.3	91.8	
11	6	99.3	108.0	94.1	129.3	134.9	104.1	
11	7	94.5	82.4	87.2	96.5	215.9	115.3	
12	1	185.6	481.4	181.7	447.6	157.1	155.1	
12	2	181.7	581.8	208.3	517.9	206.3	178.6	
12	3	122.9	160.5	117.1	165.6	274.6	152.0	
12	4	108.4	118.9	107.3	132.4	168.3	120.4	
12	5	115.4	131.4	101.3	150.8	166.7	108.2	
12	6	429.7	999.9	999.9	483.0	999.9	207.1	.85 G(0,1)+.15 G(0,3.16)
12	7	176.7	434.8	151.4	205.2	999.9	999.9	

TABLE 16

							F(w)
13 1	192.7	282.4	187.7	282.6	228.6	141.8	
13 2	108.4	130.0	118.2	139.3	177.8	128.6	
13 3	95.0	135.6	86.8	143.0	182.5	105.1	
13 4	116.2	256.5	113.8	246.0	76.2	117.3	.95 G(0,1)+.05 G(0,10)
13 5	101.1	117.7	86.6	136.8	117.5	96.9	
13 6	92.3	98.5	88.3	116.4	150.8	110.2	
13 7	101.2	98.5	95.2	123.1	144.4	120.4	
14 1	269.3	505.6	288.3	450.3	282.5	231.6	
14 2	114.0	128.6	111.9	124.1	207.9	122.4	
14 3	101.8	89.0	111.6	100.5	257.1	135.7	
14 4	117.5	193.3	99.4	204.4	127.0	104.1	
14 5	124.7	157.8	135.4	157.2	147.6	98.0	
14 6	110.0	110.0	104.0	137.3	139.7	85.7	
14 7	147.3	238.5	151.7	241.1	96.8	108.2	.9 G(0,1)+.1 G(0,10)
15 1	130.1	127.1	120.2	147.6	195.2	120.4	
15 2	117.5	200.2	193.0	138.6	198.4	111.2	
15 3	96.4	106.6	107.3	116.4	168.3	128.6	
15 4	95.9	127.2	98.0	135.9	176.2	120.4	.97 G(0,1)+.03 G(0,10)
15 5	95.2	118.2	92.1	130.4	114.3	98.0	
15 6	107.2	140.4	133.8	140.8	150.8	134.7	
15 7	110.0	198.4	128.6	192.1	82.5	129.6	
16 1	100.0	100.0	100.0	100.0	100.0	100.0	
16 2	98.2	96.9	97.1	94.3	107.9	98.0	
16 3	101.5	121.2	120.2	99.7	93.7	109.2	
16 4	110.0	158.5	165.5	106.4	112.7	136.7	G(0,1)
16 5	106.1	147.5	149.5	99.0	111.1	121.4	
16 6	100.9	117.5	114.5	103.0	84.1	116.3	
16 7	100.0	100.0	100.0	100.0	100.0	100.0	

TABLE 17

17	1	213.3	189.8	169.7	231.2	263.5	179.6	F(w)
17	2	213.0	190.0	169.9	231.0	261.9	178.6	
17	3	273.1	559.9	244.2	541.0	271.4	204.1	
17	4	439.7	999.9	498.0	999.9	560.3	381.6	
17	5	247.1	394.8	172.7	400.2	255.6	189.8	G(0, 3.16)
17	6	249.5	334.2	243.9	346.5	250.8	249.0	
17	7	213.3	189.8	169.7	231.2	263.5	179.6	

18	1	290.4	271.3	185.0	350.1	671.4	595.9	
18	2	290.5	271.3	185.0	350.3	671.4	595.9	
18	3	294.0	304.6	182.8	377.4	671.4	591.8	
18	4	263.6	241.3	207.2	307.6	677.8	591.8	
18	5	252.6	206.1	162.0	282.6	701.6	576.5	G(0,10)
18	6	319.3	386.2	255.8	414.3	679.4	636.7	
18	7	290.4	271.3	185.0	350.1	671.4	595.9	

19	1	157.9	167.8	200.0	156.5	127.0	239.8	
19	2	140.3	195.6	230.3	126.6	95.2	200.0	
19	3	131.0	227.3	248.1	116.4	93.7	203.1	
19	4	105.0	96.4	133.8	101.0	109.5	155.1	
19	5	97.1	77.7	116.0	95.4	100.0	118.4	.8 G(0,1)+.2 L(0,3.16)
19	6	134.8	249.3	282.8	106.4	101.6	163.3	
19	7	140.3	161.9	194.9	135.2	111.1	187.8	

20	1	135.6	135.9	122.4	146.4	181.0	155.1	
20	2	146.6	188.4	164.2	191.6	174.6	128.6	
20	3	179.5	372.6	350.1	243.5	347.6	130.6	
20	4	138.1	213.6	130.5	220.4	146.0	117.3	
20	5	129.9	152.5	116.7	172.8	192.1	117.3	.8 G(0,1)+.2 G(0,3.16)
20	6	141.2	211.0	125.9	228.0	150.8	121.4	
20	7	161.1	257.0	149.9	253.3	263.5	146.9	

COMPUTER PRINTOUTS

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000

Nominal noise:          Gaussian         Mean= 0.00        Std. dev.= 1.00

Contaminating noise: Laplacian           Mean= 0.00        Std. dev.= 3.16

Contamination level =     0.10

The Hampel parameters are:             s1  = 1.00              s2 = 1.00
                                       d1  = 0.17e 38          d2 = 0.17e 38
                                          ucutof = 0.17e 38


Stages run  300  Error    =  0.4467
      Std. dev. of error=  0.2669

Parameter    Std. dev.s of
estimates      estimates
  -1.572         0.485
   0.766         0.671
   0.964         0.213
   0.453         0.580


Stages run  600  Error    =  0.3826
      Std. dev. of error=  0.3025

Parameter    Std. dev.s of
estimates      estimates
  -1.543         0.452
   0.706         0.616
   1.007         0.197
   0.490         0.570


Stages run  900  Error    =  0.2640
      Std. dev. of error=  0.1781

Parameter    Std. dev.s of
estimates      estimates
  -1.478         0.280
   0.664         0.411
   0.999         0.174
   0.554         0.350


Stages run 1200  Error    =  0.2156
```

Std. dev. of error= 0.1476

Parameter estimates	Std. dev.s of estimates
-1.497	0.233
0.689	0.339
1.004	0.126
0.526	0.294

Stages run 1500 Error = 0.2106
Std. dev. of error= 0.1540

Parameter estimates	Std. dev.s of estimates
-1.519	0.232
0.704	0.332
0.996	0.110
0.497	0.309

Stages run 1800 Error = 0.1935
Std. dev. of error= 0.1781

Parameter estimates	Std. dev.s of estimates
-1.536	0.239
0.726	0.323
0.993	0.101
0.477	0.319

Stages run 2100 Error = 0.1818
Std. dev. of error= 0.1633

Parameter estimates	Std. dev.s of estimates
-1.519	0.214
0.705	0.313
0.999	0.086
0.496	0.294

Stages run 2400 Error = 0.1594
Std. dev. of error= 0.1072

Parameter estimates	Std. dev.s of estimates
-1.521	0.181
0.702	0.235
1.009	0.087
0.479	0.226

Stages run 2700 Error = 0.1446
Std. dev. of error= 0.1034

Parameter estimates	Std. dev.s of estimates
-1.539	0.155
0.719	0.221
1.004	0.082
0.464	0.208

Stages run 3000 Error = 0.1342
Std. dev. of error= 0.0712

Parameter estimates	Std. dev.s of estimates
-1.526	0.138
0.703	0.185
1.005	0.078
0.484	0.178

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
   -1.500
    0.700
    0 000
    1.000
    0.500
    0.000

Nominal noise:          Gaussian        Mean= 0.00          Std. dev.= 1.00

Contaminating noise: Laplacian          Mean= 0.00          Std. dev.= 3.16

Contamination level =     0.10

The Hampel parameters are:          s1 = 1.00            s2 = 0.00
                                    d1 = 0.20e 01        d2 = 0.17e 38
                                        ucutof = 0.17e 38


Stages run  300  Error    =  0.4106
       Std. dev. of error=  0.3293

Parameter    Std. dev.s of
estimates      estimates
   -1.581        0.486
    0.777        0.622
    0.947        0.221
    0.427        0.642


Stages run  600  Error    =  0.3864
       Std. dev. of error=  0.4038

Parameter    Std. dev.s of
estimates      estimates
   -1.582        0.518
    0.773        0.655
    0.998        0.198
    0.424        0.701


Stages run  900  Error    =  0.2711
       Std. dev. of error=  0.2059

Parameter    Std. dev.s of
estimates      estimates
   -1.534        0.310
    0.720        0.419
    0.990        0.173
    0.497        0.399


Stages run 1200  Error    =  0.2177
```

Std. dev. of error= 0.1475

Parameter estimates	Std. dev.s of estimates
-1.514	0.231
0.711	0.343
0.995	0.131
0.512	0.295

Stages run 1500 Error = 0.1993
Std. dev. of error= 0.1118

Parameter estimates	Std. dev.s of estimates
-1.526	0.211
0.701	0.289
0.988	0.117
0.506	0.257

Stages run 1800 Error = 0.1699
Std. dev. of error= 0.1251

Parameter estimates	Std. dev.s of estimates
-1.529	0.191
0.712	0.262
0.987	0.106
0.494	0.245

Stages run 2100 Error = 0.1655
Std. dev. of error= 0.1215

Parameter estimates	Std. dev.s of estimates
-1.507	0.175
0.703	0.269
0.992	0.093
0.510	0.237

Stages run 2400 Error = 0.1488
Std. dev. of error= 0.0909

Parameter estimates	Std. dev.s of estimates
-1.511	0.154
0.699	0.223
1.001	0.091
0.495	0.198

Stages run 2700 Error = 0.1348
Std. dev. of error= 0.0784

Parameter estimates	Std. dev.s of estimates
-1.528	0.133
0.713	0.202
0.999	0.084
0.479	0.174

```
Stages run 3000  Error       =  0.1228
        Std. dev. of error=  0.0595
```

Parameter estimates	Std. dev.s of estimates
-1.519	0.118
0.705	0.173
1.002	0.081
0.488	0.153

Jun 6 01:02 noise5 Page 1

```
System order n =  2

The system parameters are(in the ordera1,..,an,b1,..,bn) :
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000

Nominal noise:        Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise: Laplacian        Mean= 0.00        Std. dev.= 3.16

Contamination level =    0.10

The Hampel parameters are:          s1 = 1.00             s2 = 0.00
                                    d1 = 0.15e 01         d2 = 0.17e 38
                                        ucutof = 0.17e 38


Stages run  300  Error    =  0.3865
       Std. dev. of error=  0.3730

Parameter    Std. dev.s of
estimates      estimates
  -1.561        0.548
   0.735        0.588
   0.930        0.206
   0.452        0.671


Stages run  600  Error    =  0.3909
       Std. dev. of error=  0.4879

Parameter    Std. dev.s of
estimates      estimates
  -1.587        0.607
   0.773        0.689
   0.985        0.193
   0.414        0.812


Stages run  900  Error    =  0.2811
       Std. dev. of error=  0.2415

Parameter    Std. dev.s of
estimates      estimates
  -1.534        0.347
   0.727        0.438
   0.983        0.175
   0.488        0.450


Stages run 1200  Error    =  0.2349
```

```
        Std. dev. of error=  0.1841

Parameter    Std. dev.s of
estimates      estimates
  -1.518         0.263
   0.719         0.382
   0.989         0.138
   0.502         0.347


Stages run 1500  Error    =  0.2117
        Std. dev. of error=  0.1137

Parameter    Std. dev.s of
estimates      estimates
  -1.518         0.215
   0.696         0.301
   0.982         0.124
   0.512         0.279


Stages run 1800  Error    =  0.1788
        Std. dev. of error=  0.1100

Parameter    Std. dev.s of
estimates      estimates
  -1.509         0.186
   0.710         0.260
   0.982         0.113
   0.504         0.245


Stages run 2100  Error    =  0.1698
        Std. dev. of error=  0.1087

Parameter    Std. dev.s of
estimates      estimates
  -1.503         0.181
   0.688         0.255
   0.986         0.101
   0.521         0.230


Stages run 2400  Error    =  0.1532
        Std. dev. of error=  0.0936

Parameter    Std. dev.s of
estimates      estimates
  -1.505         0.164
   0.692         0.223
   0.995         0.099
   0.506         0.205


Stages run 2700  Error    =  0.1408
        Std. dev. of error=  0.0767
```

Parameter estimates	Std. dev.s of estimates
-1.518	0.140
0.703	0.202
0.994	0.092
0.491	0.182

Stages run 3000 Error = 0.1302
Std. dev. of error= 0.0602

Parameter estimates	Std. dev.s of estimates
-1.516	0.122
0.690	0.181
0.997	0.089
0.499	0.162

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000

Nominal noise:          Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise: Laplacian          Mean= 0.00        Std. dev.= 3.16

Contamination level =    0.10

The Hampel parameters are:          s1 = 1.00            s2 = 0.00
                                    d1 = 0.10e 01        d2 = 0.17e 38
                                        ucutof = 0.17e 38


Stages run  300  Error    =  0.4494
       Std. dev. of error=  0.5282

Parameter     Std. dev.s of
estimates       estimates
  -1.568          0.654
   0.788          0.804
   0.901          0.204
   0.407          0.879


Stages run  600  Error    =  0.4169
       Std. dev. of error=  0.5837

Parameter     Std. dev.s of
estimates       estimates
  -1.610          0.788
   0.780          0.724
   0.951          0.181
   0.392          0.918


Stages run  900  Error    =  0.3331
       Std. dev. of error=  0.3521

Parameter     Std. dev.s of
estimates       estimates
  -1.553          0.436
   0.758          0.587
   0.955          0.169
   0.456          0.603


Stages run 1200  Error    =  0.2747
```

```
        Std. dev. of error=  0.2423

Parameter     Std. dev.s of
estimates       estimates
  -1.534          0.354
   0.733          0.447
   0.966          0.143
   0.486          0.432


Stages run 1500  Error    =  0.2501
        Std. dev. of error=  0.1532

Parameter     Std. dev.s of
estimates       estimates
  -1.512          0.265
   0.713          0.366
   0.961          0.132
   0.505          0.346


Stages run 1800  Error    =  0.2164
        Std. dev. of error=  0.1213

Parameter     Std. dev.s of
estimates       estimates
  -1.507          0.223
   0.709          0.314
   0.961          0.119
   0.508          0.285


Stages run 2100  Error    =  0.2008
        Std. dev. of error=  0.1191

Parameter     Std. dev.s of
estimates       estimates
  -1.507          0.216
   0.683          0.295
   0.967          0.111
   0.524          0.264


Stages run 2400  Error    =  0.1859
        Std. dev. of error=  0.1111

Parameter     Std. dev.s of
estimates       estimates
  -1.502          0.184
   0.694          0.276
   0.975          0.109
   0.509          0.253


Stages run 2700  Error    =  0.1681
        Std. dev. of error=  0.1006
```

Parameter estimates	Std. dev.s of estimates
-1.508	0.168
0.703	0.250
0.976	0.105
0.499	0.226

Stages run 3000 Error = 0.1595
Std. dev. of error= 0.0823

Parameter estimates	Std. dev.s of estimates
-1.511	0.157
0.686	0.227
0.979	0.101
0.509	0.203

Jun 6 01:23 noise5 Page 1

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :

-1.500
0.700
0.000
1.000
0.500
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Laplacian Mean= 0.00 Std. dev.= 3.16

Contamination level = 0.10

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.70e 00 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 0.6014
Std. dev. of error= 0.5519

Parameter estimates	Std. dev.s of estimates
-1.653	0.925
0.785	0.806
0.869	0.258
0.303	1.001

Stages run 600 Error = 1.6410
Std. dev. of error= 6.6624

Parameter estimates	Std. dev.s of estimates
-2.210	3.986
2.419	9.308
0.953	0.347
-1.169	8.885

Stages run 900 Error = 0.4481
Std. dev. of error= 0.5706

Parameter estimates	Std. dev.s of estimates
-1.597	0.663
0.817	0.856
0.926	0.201
0.352	0.915

Stages run 1200 Error = 0.3700

```
        Std. dev. of error=  0.3590

Parameter     Std. dev.s of
estimates       estimates
  -1.557          0.483
   0.762          0.610
   0.936          0.175
   0.414          0.637


Stages run 1500  Error   =   0.3269
        Std. dev. of error=  0.2212

Parameter     Std. dev.s of
estimates       estimates
  -1.528          0.370
   0.725          0.459
   0.934          0.162
   0.452          0.490


Stages run 1800  Error   =   0.2941
        Std. dev. of error=  0.1962

Parameter     Std. dev.  of
estimates       estimates
  -1.510          0.298
   0.725          0.439
   0.939          0.149
   0.462          0.435


Stages run 2100  Error   =   0.2769
        Std. dev. of error=  0.1786

Parameter     Std. dev.s of
estimates       estimates
  -1.502          0.288
   0.709          0.406
   0.944          0.144
   0.479          0.401


Stages run 2400  Error   =   0.2607
        Std. dev. of error=  0.1683

Parameter     Std. dev.s of
estimates       estimates
  -1.505          0.265
   0.706          0.378
   0.950          0.139
   0.474          0.386


Stages run 2700  Error   =   0.2390
        Std. dev. of error=  0.1541
```

Parameter estimates	Std. dev.s of estimates
-1.514	0.254
0.701	0.338
0.952	0.136
0.474	0.350

Stages run 3000 Error = 0.2261
Std. dev. of error= 0.1371

Parameter estimates	Std. dev.s of estimates
-1.505	0.234
0.693	0.317
0.956	0.129
0.484	0.324

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000

Nominal noise:          Gaussian         Mean= 0.00          Std. dev.= 1.00

Contaminating noise: Laplacian           Mean= 0.00          Std. dev.= 3.16

Contamination level =     0.10

The Hampel parameters are:            s1  = 1.00               s2 =-0.27
                                      d1  = 0.12e 01           d2  = 0.35e 01
                                          ucutof = 0.79e 01


Stages run  300  Error    =  0.3992
       Std. dev. of error=  0.4821

Parameter     Std. dev.s of
estimates       estimates
  -1.570          0.605
   0.769          0.699
   0.913          0.198
   0.426          0.804


Stages run  600  Error    =  0.4009
       Std. dev. of error=  0.5806

Parameter     Std. dev.s of
estimates       estimates
  -1.604          0.701
   0.798          0.760
   0.965          0.183
   0.392          0.922


Stages run  900  Error    =  0.2974
       Std  dev. of error=  0.2746

Parameter     Std. dev.s of
estimates       estimates
  -1.557          0.428
   0.733          0.447
   0.968          0.172
   0.480          0.486


Stages run 1200  Error    =  0.2493
```

```
        Std. dev. of error=  0.2184

Parameter     Std. dev.s of
estimates       estimates
  -1.538          0.309
   0.720          0.412
   0.975          0.140
   0.498          0.388


Stages run 1500  Error    =  0.2271
        Std. dev. of error=  0.1342

Parameter     Std. dev.s of
estimates       estimates
  -1.521          0.246
   0.703          0.324
   0.970          0.128
   0.512          0.307


Stages run 1800  Error    =  0.1931
        Std. dev. of error=  0.1058

Parameter     Std. dev.s of
estimates       estimates
  -1.523          0.204
   0.698          0.271
   0.971          0.117
   0.512          0.251


Stages run 2100  Error    =  0.1809
        Std. dev. of error=  0.1055

Parameter     Std. dev.s of
estimates       estimates
  -1.505          0.189
   0.688          0.267
   0.975          0.107
   0.526          0.234


Stages run 2400  Error    =  0.1650
        Std. dev. of error=  0.0998

Parameter     Std. dev.s of
estimates       estimates
  -1.505          0.171
   0.693          0.246
   0.983          0.105
   0.512          0.218


Stages run 2700  Error    =  0.1461
        Std. dev. of error=  0.0879
```

Parameter estimates	Std. dev.s of estimates
-1.516	0.151
0.698	0.215
0.984	0.098
0.500	0.192

Stages run 3000 Error = 0.1378
Std. dev. of error= 0.0707

Parameter estimates	Std. dev.s of estimates
-1.511	0.136
0.699	0.196
0.987	0.095
0.508	0.171

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :

```
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000
```

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.0(

Contaminating noise: Laplacian Mean= 0.00 Std. dev.= 3.1(

Contamination level = 0.10

The Hampel parameters are: s1 = 1.00 s2 =-0.50
d1 = 0.25e 01 d2 = 0.45e 01
ucutof = 0.95e 01

Stages run 300 Error = 0.4293
Std. dev. of error= 0.3185

Parameter estimates	Std. dev.s of estimates
-1.577	0.444
0.811	0.660
0.956	0.227
0.408	0.653

Stages run 600 Error = 0.3795
Std. dev. of error= 0.3367

Parameter estimates	Std. dev.s of estimates
-1.551	0.447
0.774	0.627
1.004	0.207
0.444	0.617

Stages run 900 Error = 0.2621
Std. dev. of error= 0.1810

Parameter estimates	Std. dev.s of estimates
-1.535	0.295
0.697	0.398
0.992	0.172
0.515	0.358

Stages run 1200 Error = 0.2068

Std. dev. of error= 0.1307

Parameter estimates	Std. dev.s of estimates
-1.506	0.210
0.705	0.328
0.995	0.129
0.523	0.265

Stages run 1500 Error = 0.1916
Std. dev. of error= 0.1094

Parameter estimates	Std. dev.s of estimates
-1.517	0.196
0.708	0.286
0.988	0.113
0.508	0.247

Stages run 1800 Error = 0.1666
Std. dev. of error= 0.1337

Parameter estimates	Std. dev.s of estimates
-1.523	0.185
0.728	0.271
0.968	0.103
0.490	0.249

Stages run 2100 Error = 0.1629
Std. dev. of error= 0.1214

Parameter estimates	Std. dev.s of estimates
-1.523	0.184
0.692	0.262
0.992	0.087
0.512	0.232

Stages run 2400 Error = 0.1481
Std. dev. of error= 0.0889

Parameter estimates	Std. dev.s of estimates
-1.523	0.152
0.692	0.228
1.001	0.086
0.498	0.190

Stages run 2700 Error = 0.1313
Std. dev. of error= 0.0802

Parameter estimates	Std. dev.s of estimates
-1.524	0.136
0.720	0.199
0.999	0.080
0.478	0.169

```
Stages run 3000  Error     =  0.1191
       Std. dev. of error=  0.0616
```

Parameter estimates	Std. dev.s of estimates
-1.522	0.116
0.706	0.171
1.002	0.078
0.489	0.149

```
System order n =  2

The system parameters are(in the ordera1,..,an,b1,...,bn) :
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000

Nominal noise:          Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise:    Gaussian        Mean= 0.00        Std. dev.=10.00

Contamination level =      0.05

The Hampel parameters are:           s1  = 1.00           s2 = 1.00
                                     d1  = 0.17e 38       d2 = 0.17e 38
                                         ucutof = 0.17e 38


Stages run  300  Error    =  1.2217
       Std. dev. of error=  2.4015

Parameter     Std. dev.s of
estimates       estimates
  -2.382          3.175
   1.270          3.039
   1.079          0.326
  -0.124          2.836


Stages run  600  Error    =  0.9287
       Std. dev. of error=  1.1874

Parameter     Std. dev.s of
estimates       estimates
  -1.823          1.370
   0.990          2.104
   1.110          0.223
   0.069          1.526


Stages run  900  Error    =  0.6420
       Std. dev. of error=  0.6132

Parameter     Std. dev.s of
estimates       estimates
  -1.885          0.755
   1.174          1.002
   1.061          0.183
   0.029          0.968


Stages run 1200  Error    =  0.5569
```

Std. dev. of error= 0.5593

Parameter estimates	Std. dev.s of estimates
-1.771	0.689
1.060	0.911
1.027	0.172
0.180	0.919

Stages run 1500 Error = 0.4093
Std. dev. of error= 0.4135

Parameter estimates	Std. dev.s of estimates
-1.694	0.508
0.936	0.635
1.024	0.190
0.292	0.719

Stages run 1800 Error = 0.3186
Std. dev. of error= 0.2205

Parameter estimates	Std. dev.s of estimates
-1.613	0.332
0.822	0.420
0.999	0.145
0.424	0.508

Stages run 2100 Error = 0.2767
Std. dev. of error= 0.2118

Parameter estimates	Std. dev.s of estimates
-1.613	0.279
0.866	0.390
0.995	0.123
0.399	0.435

Stages run 2400 Error = 0.2645
Std. dev. of error= 0.1912

Parameter estimates	Std. dev.s of estimates
-1.608	0.279
0.834	0.387
0.987	0.121
0.422	0.384

Stages run 2700 Error = 0.2276
Std. dev. of error= 0.1597

Parameter estimates	Std. dev.s of estimates
-1.609	0.222
0.825	0.309
0.993	0.115
0.408	0.339

Stages run 3000 Error = 0.2169
Std. dev. of error= 0.1395

Parameter estimates	Std. dev.s of estimates
-1.580	0.211
0.797	0.294
0.994	0.115
0.426	0.316

System order n = 2

The system parameters are(in the ordera1,..,an,b1,..,bn) :
-1.500
0.700
0.000
1.000
0.500
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Gaussian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.05

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.20e 01 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 0.5110
Std. dev. of error= 0.5308

Parameter estimates	Std. dev.s of estimates
-1.772	0.600
1.064	0.952
1.122	0.278
0.191	0.711

Stages run 600 Error = 0.3420
Std. dev. of error= 0.2418

Parameter estimates	Std. dev.s of estimates
-1.649	0.317
0.896	0.508
1.099	0.193
0.277	0.429

Stages run 900 Error = 0.3069
Std. dev. of error= 0.2161

Parameter estimates	Std. dev.s of estimates
-1.668	0.297
0.858	0.420
1.059	0.159
0.296	0.416

Stages run 1200 Error = 0.2605

```
        Std. dev. of error=  0.1256

Parameter     Std. dev.s of
estimates       estimates
  -1.619          0.236
   0.789          0.312
   1.034          0.142
   0.385          0.352


Stages run 1500  Error    =  0.2271
        Std. dev. of error=  0.1397

Parameter     Std. dev.s of
estimates       estimates
  -1.583          0.208
   0.772          0.274
   1.026          0.152
   0.413          0.349


Stages run 1800  Error    =  0.1957
        Std. dev. of error=  0.1243

Parameter     Std. dev.s of
estimates       estimates
  -1.555          0.198
   0.745          0.238
   1.011          0.121
   0.457          0.311


Stages run 2100  Error    =  0.1818
        Std. dev. of error=  0.1249

Parameter     Std. dev.s of
estimates       estimates
  -1.570          0.179
   0.778          0.233
   1.009          0.102
   0.427          0.284


Stages run 2400  Error    =  0.1587
        Std. dev. of error=  0.1207

Parameter     Std. dev.s of
estimates       estimates
  -1.554          0.169
   0.749          0.220
   1.000          0.103
   0.457          0.253


Stages run 2700  Error    =  0.1409
        Std. dev. of error=  0.1096
```

Parameter estimates	Std. dev.s of estimates
-1.552	0.142
0.753	0.196
1.004	0.090
0.450	0.229

Stages run 3000 Error = 0.1273
Std. dev. of error= 0.0841

Parameter estimates	Std. dev.s of estimates
-1.543	0.125
0.725	0.161
1.002	0.087
0.467	0.201

Jun 7 02:00 noise13 Page 1

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
-1.500
0.700
0.000
1.000
0.500
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Gaussian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.05

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.15e 01 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 0.4773
Std. dev. of error= 0.4407

Parameter estimates	Std. dev.s of estimates
-1.739	0.537
1.009	0.788
1.147	0.310
0.187	0.637

Stages run 600 Error = 0.3262
Std. dev. of error= 0.2276

Parameter estimates	Std. dev.s of estimates
-1.646	0.310
0.868	0.471
1.112	0.210
0.275	0.395

Stages run 900 Error = 0.3042
Std. dev. of error= 0.1995

Parameter estimates	Std. dev.s of estimates
-1.647	0.287
0.861	0.396
1.076	0.172
0.280	0.396

Stages run 1200 Error = 0.2617

Std. dev. of error= 0.1242

Parameter estimates	Std. dev.s of estimates
-1.609	0.223
0.798	0.307
1.052	0.156
0.359	0.350

Stages run 1500 Error = 0.2334
Std. dev. of error= 0.1247

Parameter estimates	Std. dev.s of estimates
-1.590	0.215
0.772	0.251
1.042	0.162
0.391	0.342

Stages run 1800 Error = 0.2058
Std. dev. of error= 0.1233

Parameter estimates	Std. dev.s of estimates
-1.575	0.209
0.737	0.234
1.026	0.131
0.434	0.319

Stages run 2100 Error = 0.1934
Std. dev. of error= 0.1291

Parameter estimates	Std. dev.s of estimates
-1.578	0.189
0.772	0.234
1.023	0.115
0.408	0.303

Stages run 2400 Error = 0.1659
Std. dev. of error= 0.1285

Parameter estimates	Std. dev.s of estimates
-1.557	0.172
0.747	0.226
1.013	0.113
0.439	0.270

Stages run 2700 Error = 0.1462
Std. dev. of error= 0.1159

Parameter estimates	Std. dev.s of estimates
-1.550	0.142
0.750	0.201
1.016	0.098
0.437	0.244

Stages run 3000 Error = 0.1329
Std. dev. of error= 0.0944

Parameter estimates	Std. dev.s of estimates
-1.546	0.136
0.723	0.164
1.012	0.094
0.453	0.216

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
   -1.500
    0.700
    0.000
    1.000
    0.500
    0.000

Nominal noise:          Gaussian          Mean= 0.00          Std. dev.= 1.00

Contaminating noise:    Gaussian          Mean= 0.00          Std. dev.=10.00

Contamination level =      0.05

The Hampel parameters are:              s1 = 1.00             s2 = 0.00
                                        d1 = 0.10e 01         d2 = 0.17e 38
                                           ucutof = 0.17e 38


Stages run  300  Error    =  0.5124
       Std. dev. of error=  0.4897

Parameter     Std. dev.s of
estimates       estimates
   -1.758         0.568
    1.002         0.781
    1.156         0.344
    0.093         0.778


Stages run  600  Error    =  0.3307
       Std. dev. of error=  0.2061

Parameter     Std. dev.s of
estimates       estimates
   -1.665         0.310
    0.826         0.438
    1.123         0.246
    0.248         0.368


Stages run  900  Error    =  0.3141
       Std. dev. of error=  0.2022

Parameter     Std. dev.s of
estimates       estimates
   -1.638         0.264
    0.878         0.398
    1.100         0.209
    0.228         0.388


Stages run 1200  Error    =  0.2827
```

Std. dev. of error= 0.1551

Parameter estimates	Std. dev.s of estimates
-1.620	0.230
0.829	0.319
1.079	0.191
0.286	0.375

Stages run 1500 Error = 0.2584
Std. dev. of error= 0.1509

Parameter estimates	Std. dev.s of estimates
-1.593	0.211
0.818	0.280
1.066	0.187
0.316	0.372

Stages run 1800 Error = 0.2270
Std. dev. of error= 0.1361

Parameter estimates	Std. dev.s of estimates
-1.576	0.194
0.791	0.249
1.054	0.162
0.356	0.340

Stages run 2100 Error = 0.2132
Std. dev. of error= 0.1454

Parameter estimates	Std. dev.s of estimates
-1.608	0.205
0.781	0.240
1.047	0.142
0.351	0.321

Stages run 2400 Error = 0.1865
Std. dev. of error= 0.1421

Parameter estimates	Std. dev.s of estimates
-1.580	0.184
0.765	0.228
1.038	0.137
0.377	0.296

Stages run 2700 Error = 0.1698
Std. dev. of error= 0.1305

Parameter estimates	Std. dev.s of estimates
-1.580	0.158
0.757	0.211
1.039	0.123
0.384	0.271

Stages run 3000 Error = 0.1582
Std. dev. of error= 0.1220

Parameter estimates	Std. dev.s of estimates
-1.560	0.151
0.755	0.194
1.035	0.120
0.394	0.256

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :

```
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000
```

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Gaussian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.05

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.70e 00 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 0.7465
Std. dev. of error= 0.8585

Parameter estimates	Std. dev.s of estimates
-1.826	1.024
0.537	1.519
1.124	0.336
0.328	1.232

Stages run 600 Error = 0.5207
Std. dev. of error= 0.5520

Parameter estimates	Std. dev.s of estimates
-1.778	0.572
1.042	0.771
1.132	0.303
-0.037	0.885

Stages run 900 Error = 0.4167
Std. dev. of error= 0.3055

Parameter estimates	Std. dev.s of estimates
-1.769	0.385
0.923	0.513
1.106	0.240
0.077	0.533

Stages run 1200 Error = 0.3714

```
        Std. dev. of error=  0.2862

Parameter     Std. dev.s of
estimates      estimates
  -1.702          0.361
   0.912          0.429
   1.096          0.230
   0.128          0.526


Stages run 1500  Error    =  0.3384
        Std. dev. of error=  0.2943

Parameter     Std. dev.s of
estimates      estimates
  -1.707          0.414
   0.856          0.395
   1.078          0.209
   0.181          0.507


Stages run 1800  Error    =  0.2940
        Std. dev. of error=  0.2402

Parameter     Std. dev.s of
estimates      estimates
  -1.637          0.258
   0.860          0.358
   1.071          0.194
   0.218          0.462


Stages run 2100  Error    =  0.2814
        Std. dev. of error=  0.2351

Parameter     Std. dev.s of
estimates      estimates
  -1.673          0.292
   0.827          0.345
   1.063          0.174
   0.233          0.425


Stages run 2400  Error    =  0.2464
        Std. dev. of error=  0.2026

Parameter     Std. dev.s of
estimates      estimates
  -1.639          0.260
   0.808          0.275
   1.058          0.169
   0.264          0.380


Stages run 2700  Error    =  0.2349
        Std. dev. of error=  0.1919
```

Parameter estimates	Std. dev.s of estimates
-1.615	0.225
0.823	0.271
1.059	0.163
0.269	0.362

```
Stages run 3000  Error        =  0.2218
        Std. dev. of error=  0.1826
```

Parameter estimates	Std. dev.s of estimates
-1.614	0.225
0.804	0.260
1.054	0.154
0.285	0.339

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000

Nominal noise:         Gaussian         Mean= 0.00        Std. dev.= 1.00

Contaminating noise:  Gaussian         Mean= 0.00        Std. dev.=10.00

Contamination level =    0.05

The Hampel parameters are:            s1  = 1.00              s2 =-0.27
                                      d1  = 0.12e 01          d2  = 0.35e 01
                                          ucutof = 0.79e 01


Stages run  300  Error    =  0.5508
        Std. dev. of error=  0.5041

Parameter    Std. dev.s of
estimates      estimates
  -1.764          0.620
   1.028          0.921
   1.170          0.329
   0.120          0.729


Stages run  600  Error    =  0.3782
        Std. dev. of error=  0.4002

Parameter    Std. dev.s of
estimates      estimates
  -1.656          0.470
   0.879          0.726
   1.118          0.227
   0.260          0.531


Stages run  900  Error    =  0.3275
        Std. dev. of error=  0.2703

Parameter    Std. dev.s of
estimates      estimates
  -1.639          0.359
   0.860          0.501
   1.090          0.191
   0.271          0.443


Stages run 1200  Error    =  0.2788
```

```
        Std. dev. of error=  0.1577

Parameter     Std. dev.s of
estimates       estimates
  -1.626          0.268
   0.789          0.350
   1.068          0.171
   0.338          0.363


Stages run 1500  Error    =  0.2450
        Std. dev. of error=  0.1371

Parameter     Std. dev.s of
estimates       estimates
  -1.586          0.220
   0.770          0.292
   1.054          0.169
   0.370          0.346


Stages run 1800  Error    =  0.2216
        Std. dev. of error=  0.1276

Parameter     Std. dev.s of
estimates       estimates
  -1.558          0.206
   0.760          0.266
   1.044          0.148
   0.401          0.327


Stages run 2100  Error    =  0.2119
        Std. dev. of error=  0.1434

Parameter     Std. dev.s of
estimates       estimates
  -1.577          0.213
   0.772          0.271
   1.038          0.128
   0.385          0.317


Stages run 2400  Error    =  0.1856
        Std. dev. of error=  0.1373

Parameter     Std. dev.s of
estimates       estimates
  -1.551          0.181
   0.752          0.260
   1.030          0.125
   0.415          0.289


Stages run 2700  Error    =  0.1698
        Std. dev. of error=  0.1195
```

Parameter estimates	Std. dev.s of estimates
-1.546	0.156
0.747	0.236
1.031	0.109
0.419	0.262

Stages run 3000 Error = 0.1553
Std. dev. of error= 0.1033

Parameter estimates	Std. dev.s of estimates
-1.555	0.165
0.714	0.198
1.026	0.102
0.437	0.233

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.700
   0.000
   1.000
   0.500
   0.000

Nominal noise:        Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise:  Gaussian        Mean= 0.00        Std. dev.=10.00

Contamination level =     0.05

The Hampel parameters are:          s1 = 1.00           s2 =-0.50
                                    d1 = 0.25e 01       d2 = 0.45e 01
                                       ucutof = 0.95e 01


Stages run  300  Error    =  0.5347
       Std. dev. of error=  0.5141

Parameter    Std. dev.s of
estimates      estimates
  -1.764         0.628
   1.085         0.919
   1.124         0.278
   0.179         0.737


Stages run  600  Error    =  0.3368
       Std. dev. of error=  0.3118

Parameter    Std. dev.s of
estimates      estimates
  -1.663         0.417
   0.846         0.564
   1.084         0.188
   0.323         0.475


Stages run  900  Error    =  0.3111
       Std. dev. of error=  0.2573

Parameter    Std. dev.s of
estimates      estimates
  -1.647         0.364
   0.847         0.457
   1.049         0.159
   0.326         0.456


Stages run 1200  Error    =  0.2512
```

```
        Std. dev. of error=  0.1187

Parameter     Std. dev.s of
estimates       estimates
  -1.596          0.246
   0.761          0.298
   1.025          0.139
   0.414          0.343


Stages run 1500  Error    =  0.2156
        Std. dev. of error=  0.1387

Parameter     Std. dev.s of
estimates       estimates
  -1.557          0.221
   0.745          0.264
   1.017          0.145
   0.449          0.338


Stages run 1800  Error    =  0.1940
        Std. dev. of error=  0.1297

Parameter     Std. dev.s of
estimates       estimates
  -1.536          0.205
   0.721          0.248
   1.004          0.122
   0.489          0.312


Stages run 2100  Error    =  0.1789
        Std. dev. of error=  0.1318

Parameter     Std. dev.s of
estimates       estimates
  -1.566          0.192
   0.750          0.245
   1.005          0.100
   0.451          0.284


Stages run 2400  Error    =  0.1613
        Std. dev. of error=  0.1191

Parameter     Std. dev.s of
estimates       estimates
  -1.542          0.175
   0.729          0.232
   0.996          0.100
   0.479          0.250


Stages run 2700  Error    =  0.1439
        Std. dev. of error=  0.1097
```

Parameter estimates	Std. dev.s of estimates
-1.545	0.153
0.734	0.208
1.000	0.088
0.469	0.228

Stages run 3000 Error = 0.1306
Std. dev. of error= 0.0836

Parameter estimates	Std. dev.s of estimates
-1.526	0.130
0.717	0.171
0.998	0.086
0.482	0.203

System order n = 3

The system parameters are(in the ordera1,...,an,b1,...,bn) :
-1.500
0.705
-0.100
0.065
0.048
-0.008

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Laplacian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.05

The Hampel parameters are: s1 = 1.00 s2 = 1.00
d1 = 0.17e 38 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 1.9036
Std. dev. of error= 1.1228

Parameter estimates	Std. dev.s of estimates
-0.408	0.921
-0.255	0.931
-0.156	0.438
0.541	1.854
0.542	1.751
0.390	1.530

Stages run 600 Error = 2.2688
Std. dev. of error= 3.7852

Parameter estimates	Std. dev.s of estimates
-0.138	1.533
-0.171	0.824
-0.195	0.350
0.315	4.372
0.286	4.063
0.259	3.497

Stages run 900 Error = 3.6052
Std. dev. of error= 10.5981

Parameter estimates	Std. dev.s of estimates
-0.497	1.989
-0.326	1.405
-0.218	0.797

2.657	11.363
2.444	10.535
2.073	8.988

Stages run 1200 Error = 1.7120
Std. dev. of error= 1.3053

Parameter estimates	Std. dev.s of estimates
-0.386	0.762
-0.246	0.527
-0.144	0.319
0.848	1.796
0.783	1.675
0.667	1.435

Stages run 1500 Error = 3.4263
Std. dev. of error= 7.3031

Parameter estimates	Std. dev.s of estimates
-1.266	3.479
-0.928	2.894
-0.422	1.038
2.902	7.383
2.693	6.869
2.280	5.787

Stages run 1800 Error = 1.5420
Std. dev. of error= 0.8114

Parameter estimates	Std. dev.s of estimates
-0.242	0.517
-0.162	0.336
-0.089	0.189
0.688	1.360
0.631	1.263
0.530	1.085

Stages run 2100 Error = 1.8256
Std. dev. of error= 1.0611

Parameter estimates	Std. dev.s of estimates
-0.416	0.553
-0.270	0.363
-0.147	0.205
1.114	1.662
1.027	1.541
0.870	1.322

Stages run 2400 Error = 4.4335
Std. dev. of error= 11.1638

Parameter estimates	Std. dev.s of estimates
-1.268	4.248
-0.812	2.820
-0.457	1.675
3.585	11.564
3.330	10.789
2.847	9.275

Stages run 2700 Error = 2.5211
Std. dev. of error= 2.2807

Parameter estimates	Std. dev.s of estimates
-0.539	1.291
-0.356	0.875
-0.214	0.522
0.986	3.149
0.913	2.940
0.779	2.529

Stages run 3000 Error = 2.8510
Std. dev. of error= 2.6604

Parameter estimates	Std. dev.s of estimates
-0.143	1.073
-0.095	0.651
-0.052	0.347
0.471	3.875
0.425	3.595
0.348	3.077

```
System order n =  3

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.705
  -0.100
   0.065
   0.048
  -0.008

Nominal noise:         Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise: Laplacian        Mean= 0.00        Std. dev.=10.00

Contamination level =     0.05

The Hampel parameters are:          s1 = 1.00            s2 = 0.00
                                    d1 = 0.20e 01        d2 = 0.17e 38
                                        ucutof = 0.17e 38


Stages run  300  Error    = 20.0082
       Std. dev. of error= 99.0224

Parameter     Std. dev.s of
estimates       estimates
  -7.187         37.782
  -4.364         22.710
  -2.406         12.237
  18.817        100.064
  18.107         96.360
  14.413         76.581


Stages run  600  Error    =  1.8467
       Std. dev. of error=  2.8810

Parameter     Std. dev.s of
estimates       estimates
  -0.477          0.763
  -0.313          0.423
  -0.174          0.192
   1.381          3.174
   1.276          2.943
   1.083          2.512


Stages run  900  Error    =  2.1003
       Std. dev. of error=  1.6520

Parameter     Std. dev.s of
estimates       estimates
  -0.371          0.692
  -0.259          0.454
  -0.146          0.240
```

1.030	2.398
0.947	2.232
0.801	1.912

Stages run 1200 Error = 2.3946
Std. dev. of error= 3.1749

Parameter estimates	Std. dev.s of estimates
-0.181	1.483
-0.118	1.054
-0.071	0.631
0.598	3.824
0.546	3.571
0.454	3.075

Stages run 1500 Error = 1.4446
Std. dev. of error= 0.4982

Parameter estimates	Std. dev.s of estimates
-0.242	0.434
-0.168	0.295
-0.103	0.169
0.563	1.116
0.515	1.045
0.430	0.900

Stages run 1800 Error = 5.3562
Std. dev. of error= 16.4406

Parameter estimates	Std. dev.s of estimates
0.320	4.058
0.085	2.045
-0.043	0.737
-2.370	17.788
-2.202	16.501
-1.882	14.078

Stages run 2100 Error = 2.4394
Std. dev. of error= 2.7189

Parameter estimates	Std. dev.s of estimates
-0.649	1.266
-0.432	0.831
-0.247	0.491
1.658	3.203
1.536	2.989
1.309	2.575

```
Stages run 2400  Error     =  1.8752
       Std. dev. of error=  2.4158

Parameter     Std. dev.s of
estimates       estimates
  -0.524          1.082
  -0.341          0.703
  -0.185          0.375
   1.377          2.642
   1.274          2.466
   1.082          2.124

Stages run 2700  Error     =  5.9423
       Std. dev. of error= 19.1609

Parameter     Std. dev.s of
estimates       estimates
   0.194          5.640
   0.053          3.481
  -0.039          1.672
  -1.893         20.524
  -1.763         19.095
  -1.510         16.340

Stages run 3000  Error     =  1.9843
       Std. dev. of error=  1.6696

Parameter     Std. dev.s of
estimates       estimates
   0.007          0.845
   0.010          0.553
   0.007          0.305
  -0.002          2.421
  -0.013          2.256
  -0.026          1.936
```

System order n = 3

The system parameters are(in the ordera1,...,an,b1,...,bn) :

```
  -1.500
   0.705
  -0.100
   0.065
   0.048
  -0.008
```

```
Nominal noise:        Gaussian         Mean= 0.00          Std. dev.= 1.00

Contaminating noise: Laplacian         Mean= 0.00          Std. dev.=10.00

Contamination level =     0.05

The Hampel parameters are:           s1 = 1.00             s2 = 0.00
                                     d1 = 0.15e 01         d2 = 0.17e 38
                                           ucutof = 0.17e 38
```

```
Stages run  300  Error    =  3.0670
       Std. dev. of error=  7.9708
```

Parameter estimates	Std. dev.s of estimates
-0.960	2.909
-0.571	1.504
-0.287	0.480
2.349	8.373
2.174	7.753
1.869	6.723

```
Stages run  600  Error    =  2.9400
       Std. dev. of error=  4.5393
```

Parameter estimates	Std. dev.s of estimates
-0.739	1.028
-0.458	0.578
-0.235	0.260
2.578	4.970
2.390	4.615
2.033	3.933

```
Stages run  900  Error    =  7.1440
       Std. dev. of error= 20.7792
```

Parameter estimates	Std. dev.s of estimates
1.619	9.421
1.045	6.269
0.567	3.579

-4.226	21.237
-3.949	19.842
-3.406	17.102

Stages run 1200 Error = 1.6158
Std. dev. of error= 0.9473

Parameter estimates	Std. dev.s of estimates
-0.392	0.492
-0.268	0.347
-0.150	0.190
0.997	1.394
0.919	1.300
0.778	1.114

Stages run 1500 Error = 1.5217
Std. dev. of error= 0.9725

Parameter estimates	Std. dev.s of estimates
-0.279	0.438
-0.181	0.283
-0.100	0.155
0.798	1.421
0.733	1.322
0.615	1.131

Stages run 1800 Error = 2.3677
Std. dev. of error= 4.4900

Parameter estimates	Std. dev.s of estimates
-0.584	1.684
-0.388	1.119
-0.217	0.612
1.630	4.825
1.509	4.497
1.283	3.866

Stages run 2100 Error = 1.4430
Std. dev. of error= 0.4389

Parameter estimates	Std. dev.s of estimates
-0.213	0.406
-0.145	0.268
-0.086	0.149
0.506	1.114
0.462	1.041
0.384	0.897

Stages run 2400 Error = 2.5381
Std. dev. of error= 4.1726

Parameter estimates	Std. dev.s of estimates
-0.137	1.883
-0.092	1.214
-0.058	0.641
0.468	4.781
0.424	4.463
0.348	3.845

Stages run 2700 Error = 2.6136
Std. dev. of error= 2.4616

Parameter estimates	Std. dev.s of estimates
-0.637	0.993
-0.419	0.631
-0.236	0.338
1.757	3.147
1.627	2.932
1.384	2.512

Stages run 3000 Error = 1.4853
Std. dev. of error= 0.5839

Parameter estimates	Std. dev.s of estimates
-0.290	0.455
-0.195	0.311
-0.111	0.186
0.715	1.148
0.657	1.073
0.551	0.923

```
System order n =  3


The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.705
  -0.100
   0.065
   0.048
  -0.008

Nominal noise:          Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise: Laplacian          Mean= 0.00        Std. dev.=10.00

Contamination level =    0.05

The Hampel parameters are:             s1 = 1.00              s2 = 0.00
                                       d1 = 0.10e 01          d2 = 0.17e 38
                                           ucutof = 0.17e 38



Stages run  300  Error    =  1.4272
       Std. dev. of error=  0.4274

Parameter    Std. dev.s of
estimates      estimates
  -0.275         0.412
  -0.209         0.294
  -0.102         0.164
   0.802         0.922
   0.734         0.852
   0.621         0.727


Stages run  600  Error    =  1.4816
       Std. dev. of error=  0.6641

Parameter    Std. dev.s of
estimates      estimates
  -0.437         0.517
  -0.299         0.352
  -0.169         0.197
   1.047         0.938
   0.965         0.877
   0.818         0.759


Stages run  900  Error    =  2.6765
       Std. dev. of error=  3.0766

Parameter    Std. dev.s of
estimates      estimates
  -0.404         1.213
  -0.263         0.808
  -0.148         0.434
```

```
    0.981          3.965
    0.904          3.692
    0.763          3.161


Stages run 1200  Error    =  6.0640
       Std. dev. of error= 14.8386

Parameter     Std. dev.s of
estimates       estimates
   0.686          5.761
   0.419          3.791
   0.199          2.063
  -2.435         15.940
  -2.279         14.866
  -1.962         12.759


Stages run 1500  Error    =  2.1818
       Std. dev. of error=  3.1740

Parameter     Std. dev.s of
estimates       estimates
  -0.370          1.043
  -0.231          0.608
  -0.127          0.297
   1.003          3.744
   0.924          3.482
   0.781          2.981


Stages run 1800  Error    =  1.8810
       Std. dev. of error=  1.1242

Parameter     Std. dev.s of
estimates       estimates
  -0.268          0.762
  -0.182          0.519
  -0.106          0.301
   0.696          1.888
   0.638          1.762
   0.536          1.520


Stages run 2100  Error    =  2.2285
       Std. dev. of error=  2.2817

Parameter     Std. dev.s of
estimates       estimates
  -0.309          0.998
  -0.209          0.645
  -0.119          0.345
   0.899          2.997
   0.827          2.793
   0.696          2.397
```

Stages run 2400 Error = 1.4725
Std. dev. of error= 0.5871

Parameter estimates	Std. dev.s of estimates
-0.332	0.397
-0.219	0.273
-0.122	0.161
0.971	0.967
0.893	0.903
0.753	0.778

Stages run 2700 Error = 2.3430
Std. dev. of error= 2.1272

Parameter estimates	Std. dev.s of estimates
-0.527	1.047
-0.341	0.707
-0.187	0.404
1.600	2.665
1.480	2.488
1.256	2.139

Stages run 3000 Error = 1.4557
Std. dev. of error= 1.2236

Parameter estimates	Std. dev.s of estimates
-0.338	0.580
-0.224	0.386
-0.126	0.220
0.874	1.485
0.805	1.386
0.679	1.192

Jun 7 21:01 noise7 Page 1

System order n = 3

The system parameters are(in the ordera1,...,an,b1,...,bn) :

-1.500
0.705
-0.100
0.065
0.048
-0.008

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Laplacian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.05

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.70e 00 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 1.6394
Std. dev. of error= 0.6627

Parameter estimates	Std. dev.s of estimates
-0.325	0.457
-0.233	0.309
-0.152	0.192
0.481	1.491
0.443	1.390
0.379	1.184

Stages run 600 Error = 1.6829
Std. dev. of error= 1.0272

Parameter estimates	Std. dev.s of estimates
-0.377	0.555
-0.276	0.378
-0.171	0.231
0.724	1.652
0.667	1.542
0.565	1.325

Stages run 900 Error = 4.0576
Std. dev. of error= 13.7168

Parameter estimates	Std. dev.s of estimates
-1.014	3.743
-0.621	2.212
-0.306	1.019

3.571	14.327
3.313	13.325
2.822	11.389

Stages run 1200 Error = 1.9298
Std. dev. of error= 1.5294

Parameter estimates	Std. dev.s of estimates
-0.313	0.734
-0.211	0.471
-0.123	0.260
0.742	2.230
0.681	2.079
0.574	1.785

Stages run 1500 Error = 8.5212
Std. dev. of error= 35.3760

Parameter estimates	Std. dev.s of estimates
-1.501	7.762
-0.782	3.991
-0.262	1.321
7.118	37.233
6.600	34.581
5.625	29.539

Stages run 1800 Error = 1.6960
Std. dev. of error= 1.3070

Parameter estimates	Std. dev.s of estimates
-0.471	0.557
-0.310	0.365
-0.174	0.207
1.282	1.584
1.185	1.476
1.005	1.267

Stages run 2100 Error = 7.2183
Std. dev. of error= 28.1430

Parameter estimates	Std. dev.s of estimates
1.005	8.143
0.653	5.301
0.410	3.165
-3.926	29.548
-3.663	27.500
-3.143	23.517

Jun 7 21:01 noise7 Page 3

```
Stages run 2400  Error      =  1.8296
        Std. dev. of error=  1.4907

Parameter      Std. dev.s of
estimates        estimates
  -0.406           0.663
  -0.273           0.421
  -0.156           0.227
   1.042           2.007
   0.961           1.870
   0.813           1.607


Stages run 2700  Error      =  5.7958
        Std. dev. of error= 20.5034

Parameter      Std. dev.s of
estimates        estimates
  -1.674           6.771
  -1.110           4.436
  -0.618           2.417
   4.928          21.141
   4.582          19.693
   3.920          16.891


Stages run 3000  Error      =  2.4496
        Std. dev. of error=  2.6744

Parameter      Std. dev.s of
estimates        estimates
  -0.184           1.031
  -0.119           0.661
  -0.066           0.355
   0.370           3.580
   0.335           3.332
   0.276           2.852
```

```
Jun 11 23:30  noise7 Page 1

System order n =  3

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.500
   0.705
  -0.100
   0.065
   0.048
  -0.008

Nominal noise:          Gaussian          Mean= 0.00          Std. dev.= 1.00

Contaminating noise: Laplacian            Mean= 0.00          Std. dev.=10.00

Contamination level =     0.05

The Hampel parameters are:             s1 = 1.00              s2 =-0.27
                                       d1 = 0.12e 01          d2 = 0.35e 01
                                           ucutof = 0.79e 01


Stages run  300  Error    =  1.5954
       Std. dev. of error=  0.9244

Parameter    Std. dev.s of
estimates      estimates
  -0.294        0.584
  -0.193        0.370
  -0.113        0.228
   0.497        1.563
   0.457        1.455
   0.381        1.242


Stages run  600  Error    = 16.8628
       Std. dev. of error= 80.7389

Parameter    Std. dev.s of
estimates      estimates
  -4.355       21.188
  -2.595       12.559
  -1.296        6.225
  16.703       83.587
  15.527       77.739
  13.261       66.429


Stages run  900  Error    =  1.5590
       Std. dev. of error=  1.2212

Parameter    Std. dev.s of
estimates      estimates
  -0.187        0.700
  -0.133        0.468
  -0.083        0.264
```

```
    0.427           1.698
    0.389           1.588
    0.321           1.369

Stages run 1200  Error    =  1.6317
      .Std. dev. of error=  1.0872

Parameter      Std. dev.s of
estimates        estimates
   -0.123          0.777
   -0.096          0.549
   -0.036          0.307
    0.383          1.647
    0.343          1.535
    0.290          1.327

Stages run 1500  Error    =  1.4127
        Std. dev. of error=  0.4632

Parameter      Std. dev.s of
estimates        estimates
   -0.250          0.405
   -0.170          0.273
   -0.101          0.156
    0.637          1.026
    0.583          0.960
    0.488          0.828

Stages run 1800  Error    = 13.8155
        Std. dev. of error= 64.0480

Parameter      Std. dev.s of
estimates        estimates
   -3.650         17.957
   -2.265         11.054
   -1.171          5.675
   13.117         66.175
   12.196         61.569
   10.427         52.690

Stages run 2100  Error    =  2.5616
        Std. dev. of error=  3.9842

Parameter      Std. dev.s of
estimates        estimates
    0.064          1.403
    0.046          0.933
    0.022          0.510
   -0.326          4.715
   -0.314          4.391
   -0.282          3.759
```

```
Stages run 2400  Error     =  1.9350
       Std. dev. of error=  1.5758

Parameter      Std. dev.s of
estimates        estimates
  -0.193          0.772
  -0.123          0.518
  -0.062          0.297
   0.533          2.302
   0.486          2.145
   0.404          1.839

Stages run 2700  Error     =  1.4697
       Std. dev. of error=  0.5250

Parameter      Std. dev.s of
estimates        estimates
  -0.302          0.379
  -0.197          0.251
  -0.109          0.144
   0.824          1.049
   0.757          0.978
   0.637          0.839

Stages run 3000  Error     =  2.1726
       Std. dev. of error=  1.6947

Parameter      Std. dev.s of
estimates        estimates
  -0.301          0.903
  -0.201          0.612
  -0.113          0.357
   0.781          2.530
   0.718          2.359
   0.604          2.026
```

System order n = 3

The system parameters are(in the ordera1,...,an,b1,...,bn) :

-1.500
0.705
-0.100
0.065
0.048
-0.008

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Laplacian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.05

The Hampel parameters are: s1 = 1.00 s2 =-0.50
d1 = 0.25e 01 d2 = 0.45e 01
ucutof = 0.95e 01

Stages run 300 Error = 2.0325
Std. dev. of error= 2.4504

Parameter estimates	Std. dev.s of estimates
-0.019	1.024
-0.161	0.517
-0.224	0*872
0.042	2.915
-0.055	2.938
-0.006	2.415

Stages run 600 Error = 1.8570
Std. dev. of error= 1.7382

Parameter estimates	Std. dev.s of estimates
-0.274	0.648
-0.202	0.403
-0.126	0.227
0.532	2.399
0.490	2.228
0.414	1.899

Stages run 900 Error = 5.5796
Std. dev. of error= 16.9777

Parameter estimates	Std. dev.s of estimates
-0.756	3.986
-0.397	1.994
-0.183	0.903

3.450	18.285
3.190	16.969
2.707	14.463

Stages run 1200 Error = 2.6864
Std. dev. of error= 5.4072

Parameter estimates	Std. dev.s of estimates
-0.068	2.151
-0.025	1.480
-0.028	0.747
0.481	6.001
0.436	5.594
0.360	4.797

Stages run 1500 Error = 1.9143
Std. dev. of error= 1.1135

Parameter estimates	Std. dev.s of estimates
-0.207	0.675
-0.132	0.465
-0.085	0.248
0.534	1.986
0.490	1.846
0.407	1.584

Stages run 1800 Error = 2.0837
Std. dev. of error= 2.8338

Parameter estimates	Std. dev.s of estimates
-0.518	1.439
-0.333	0.911
-0.193	0.547
1.318	3.152
1.217	2.931
1.033	2.529

Stages run 2100 Error = 2.1462
Std. dev. of error= 2.5302

Parameter estimates	Std. dev.s of estimates
-0.224	1.227
-0.158	0.809
-0.099	0.430
0.481	3.166
0.439	2.956
0.365	2.545

Stages run 2400 Error = 2.3889
Std. dev. of error= 2.6754

Parameter estimates	Std. dev.s of estimates
-0.581	1.102
-0.387	0.730
-0.217	0.401
1.598	3.194
1.480	2.978
1.257	2.555

Stages run 2700 Error = 2.0355
Std. dev. of error= 1.8568

Parameter estimates	Std. dev.s of estimates
-0.283	0.859
-0.182	0.577
-0.101	0.327
0.686	2.568
0.629	2.393
0.528	2.051

Stages run 3000 Error = 3.0649
Std. dev. of error= 4.0915

Parameter estimates	Std. dev.s of estimates
-0.453	1.846
-0.272	1.228
-0.128	0.715
1.847	4.750
1.705	4.431
1.442	3.811

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000

Nominal noise:          Gaussian          Mean= 0.00          Std. dev.= 1.00

Contaminating noise: Laplacian            Mean= 0.00          Std. dev.=10.00

Contamination level =     0.03

The Hampel parameters are:              s1 = 1.00              s2 = 1.00
                                        d1 = 0.17e 38          d2 = 0.17e 38
                                            ucutof = 0.17e 38


Stages run  300  Error    =  2.0169
       Std. dev. of error=  1.7098

Parameter     Std. dev.s of
estimates       estimates
  -0.227          2.600
   1.046          2.714
  -0.061          0.247
   0.216          0.521


Stages run  600  Error    =  2.2687
       Std. dev. of error=  3.1596

Parameter     Std. dev.s of
estimates       estimates
   0.463          2.043
   1.420          5.121
  -0.067          0.187
   0.064          0.442


Stages run  900  Error    =  1.4519
       Std. dev. of error=  1.1753

Parameter     Std. dev.s of
estimates       estimates
  -0.392          1.228
   1.111          2.244
  -0.084          0.131
   0.116          0.270


Stages run 1200  Error    =  1.9809
```

```
        Std. dev. of error=  3.3571

Parameter     Std. dev.s of
estimates       estimates
   0.552          5.019
   1.187          2.221
  -0.101          0.155
  -0.056          0.787


Stages run 1500  Error    =  1.0842
        Std. dev. of error=  0.5394

Parameter     Std. dev.s of
estimates       estimates
  -0.472          1.001
   0.723          1.177
  -0.087          0.105
   0.102          0.186


Stages run 1800  Error    =  1.3335
        Std. dev. of error=  1.1961

Parameter     Std. dev.s of
estimates       estimates
  -0.161          1.762
   0.966          1.515
  -0.083          0.109
   0.020          0.422


Stages run 2100  Error    =  1.3504
        Std. dev. of error=  0.9448

Parameter     Std. dev.s of
estimates       estimates
  -0.202          1.147
   1.093          1.746
  -0.092          0.097
   0.049          0.228


Stages run 2400  Error    =  1.5304
        Std. dev. of error=  1.1430

Parameter     Std. dev.s of
estimates       estimates
  -0.101          1.908
   0.368          1.665
  -0.068          0.201
   0.073          0.469


Stages run 2700  Error    =  1.2194
        Std. dev. of error=  1.1478
```

Parameter estimates	Std. dev.s of estimates
-0.273	1.761
0.395	1.328
-0.080	0.123
0.033	0.509

Stages run 3000 Error = 1.0457
Std. dev. of error= 0.8278

Parameter estimates	Std. dev.s of estimates
-0.724	1.408
0.342	1.208
-0.072	0.178
0.074	0.350

```
System order n =  2


The system parameters are(in the ordera1,...,an,b1,...,bn) :
   -1.425
    0.496
    0.000
   -0.102
    0.173
    0.000

Nominal noise:          Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise: Laplacian          Mean= 0.00        Std. dev.=10.00

Contamination level =     0.03

The Hampel parameters are:            s1 = 1.00           s2 = 0.00
                                      d1 = 0.20e 01       d2 = 0.17e 38
                                         ucutof = 0.17e 38


Stages run  300  Error    =  1.4660
        Std. dev. of error=  1.2774

Parameter    Std. dev.s of
estimates      estimates
    0.010        2.054
    0.279        1.529
   -0.114        0.162
    0.060        0.244


Stages run  600  Error    =  2.0116
        Std. dev. of error=  2.5007

Parameter    Std. dev.s of
estimates      estimates
   -0.383        2.737
    0.338        3.897
   -0.115        0.131
    0.151        0.236


Stages run  900  Error    =  0.9692
        Std. dev. of error=  0.7315

Parameter    Std. dev.s of
estimates      estimates
   -0.879        0.843
    1.015        1.451
   -0.096        0.113
    0.101        0.129


Stages run 1200  Error    =  0.9284
```

```
        Std. dev. of error=  0.8229

Parameter      Std. dev.s of
estimates        estimates
  -0.776           0.953
   1.005           1.388
  -0.119           0.102
   0.077           0.178


Stages run 1500  Error    =  0.8648
        Std. dev. of error=  0.7324

Parameter      Std. dev.s of
estimates        estimates
  -1.083           1.396
   0.808           0.878
  -0.123           0.098
   0.121           0.174


Stages run 1800  Error    =  0.7208
        Std. dev. of error=  0.4948

Parameter      Std. dev.s of
estimates        estimates
  -0.832           0.876
   0.848           0.707
  -0.102           0.058
   0.092           0.133


Stages run 2100  Error    =  0.7009
        Std. dev. of error=  0.4543

Parameter      Std. dev.s of
estimates        estimates
  -1.147           0.812
   0.789           0.880
  -0.099           0.067
   0.127           0.118


Stages run 2400  Error    =  0.6421
        Std. dev. of error=  0.3588

Parameter      Std. dev.s of
estimates        estimates
  -1.193           0.582
   0.705           0.894
  -0.106           0.050
   0.134           0.121


Stages run 2700  Error    =  0.5402
        Std. dev. of error=  0.3332
```

Parameter estimates	Std. dev.s of estimates
-1.253	0.600
0.561	0.721
-0.108	0.054
0.139	0.125

Stages run 3000 Error = 0.5233
Std. dev. of error= 0.2450

Parameter estimates	Std. dev.s of estimates
-1.225	0.555
0.589	0.633
-0.090	0.073
0.152	0.102

AD-A038 294 BALLISTIC RESEARCH LABS ABERDEEN PROVING GROUND MD F/G 20/1

ROBUST IDENTIFICATION OF LINEAR SYSTEMS.(U)

FEB 77 V D VANDELINDE

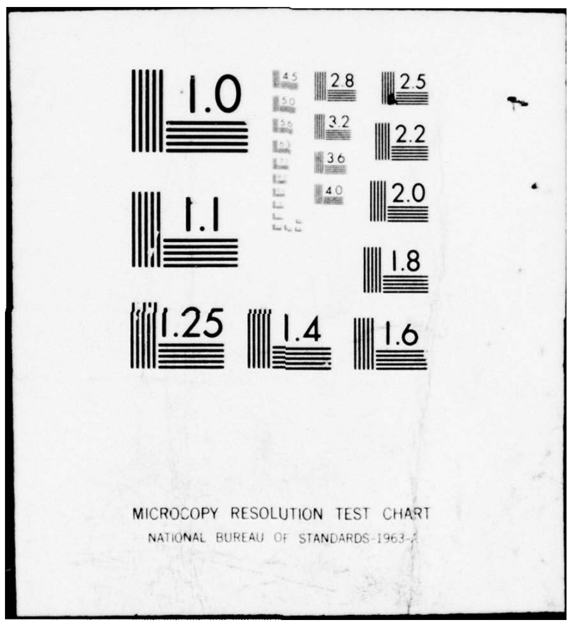
UNCLASSIFIED BRL-1960 NL

3 OF 3

AD A038294

END

DATE FILMED 5-77


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000

Nominal noise:          Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise: Laplacian          Mean= 0.00        Std. dev.=10.00

Contamination level =     0.03

The Hampel parameters are:          s1 = 1.00           s2 = 0.00
                                    d1 = 0.15e 01       d2 = 0.17e 38
                                        ucutof = 0.17e 38


Stages run  300  Error    =  1.3457
        Std. dev. of error=  1.0236

Parameter     Std. dev.s of
estimates       estimates
  -0.084          1.489
   0.289          1.577
  -0.107          0.116
   0.049          0.222


Stages run  600  Error    =  1.4969
        Std. dev. of error=  1.8338

Parameter     Std. dev.s of
estimates       estimates
  -0.295          1.917
   0.578          2.821
  -0.095          0.139
   0.137          0.219


Stages run  900  Error    =  0.8086
        Std. dev. of error=  0.2996

Parameter     Std. dev.s of
estimates       estimates
  -0.806          0.676
   0.820          0.863
  -0.089          0.116
   0.106          0.125


Stages run 1200  Error    =  0.7924
```

Std. dev. of error= 0.6257

Parameter estimates	Std. dev.s of estimates
-0.894	0.874
0.935	1.044
-0.103	0.081
0.099	0.147

Stages run 1500 Error = 0.6885
Std. dev. of error= 0.3483

Parameter estimates	Std. dev.s of estimates
-1.009	0.724
0.698	0.786
-0.098	0.069
0.128	0.129

Stages run 1800 Error = 0.6322
Std. dev. of error= 0.3401

Parameter estimates	Std. dev.s of estimates
-0.977	0.620
0.802	0.689
-0.115	0.107
0.096	0.153

Stages run 2100 Error = 0.6429
Std. dev. of error= 0.3655

Parameter estimates	Std. dev.s of estimates
-1.117	0.710
0.715	0.777
-0.102	0.068
0.124	0.104

Stages run 2400 Error = 0.5951
Std. dev. of error= 0.2805

Parameter estimates	Std. dev.s of estimates
-1.195	0.571
0.650	0.764
-0.104	0.056
0.137	0.113

Stages run 2700 Error = 0.6094
Std. dev. of error= 0.5410

Parameter estimates	Std. dev.s of estimates
-1.390	1.020
0.555	0.688
-0.108	0.071
0.147	0.099

Stages run 3000 Error = 0.4605
Std. dev. of error= 0.2328

Parameter estimates	Std. dev.s of estimates
-1.199	0.483
0.542	0.558
-0.096	0.060
0.139	0.099

System order n = 2

The system parameters are(in the orderal,...,an,b1,...,bn) :
-1.425
0.496
0.000
-0.102
0.173
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Laplacian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.03

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.10e 01 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 1.4045
Std. dev. of error= 1.2021

Parameter estimates	Std. dev.s of estimates
-0.122	1.474
0.473	1.986
-0.100	0.130
0.041	0.259

Stages run 600 Error = 1.3136
Std. dev. of error= 1.6293

Parameter estimates	Std. dev.s of estimates
-0.417	1.680
0.185	2.477
-0.078	0.172
0.148	0.218

Stages run 900 Error = 0.7768
Std. dev. of error= 0.3861

Parameter estimates	Std. dev.s of estimates
-0.898	0.726
0.805	0.905
-0.088	0.109
0.126	0.100

Stages run 1200 Error = 0.6979

Std. dev. of error= 0.5140

Parameter estimates	Std. dev.s of estimates
-1.061	0.711
0.881	0.962
-0.116	0.078
0.108	0.141

Stages run 1500 Error = 0.6606
Std. dev. of error= 0.3223

Parameter estimates	Std. dev.s of estimates
-1.156	0.606
0.652	0.874
-0.119	0.067
0.134	0.133

Stages run 1800 Error = 0.5623
Std. dev. of error= 0.3174

Parameter estimates	Std. dev.s of estimates
-1.068	0.487
0.748	0.724
-0.101	0.058
0.124	0.087

Stages run 2100 Error = 0.5848
Std. dev. of error= 0.3730

Parameter estimates	Std. dev.s of estimates
-1.115	0.602
0.665	0.783
-0.100	0.059
0.133	0.100

Stages run 2400 Error = 0.5704
Std. dev. of error= 0.2803

Parameter estimates	Std. dev.s of estimates
-1.221	0.608
0.626	0.701
-0.100	0.057
0.147	0.106

Stages run 2700 Error = 0.4983
Std. dev. of error= 0.2414

Parameter estimates	Std. dev.s of estimates
-1.282	0.550
0.516	0.612
-0.105	0.058
0.144	0.095

Stages run 3000 Error = 0.4482
Std. dev. of error= 0.2031

Parameter estimates	Std. dev.s of estimates
-1.223	0.488
0.550	0.516
-0.101	0.060
0.140	0.086

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :

```
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000
```

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Laplacian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.03

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.70e 00 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 1.6069
Std. dev. of error= 1.7903

Parameter estimates	Std. dev.s of estimates
-0.456	1.725
-0.101	2.997
-0.100	0.132
-0.009	0.327

Stages run 600 Error = 1.4854
Std. dev. of error= 1.6090

Parameter estimates	Std. dev.s of estimates
-0.484	1.965
0.505	2.509
-0.092	0.151
0.128	0.214

Stages run 900 Error = 1.1430
Std. dev. of error= 1.6198

Parameter estimates	Std. dev.s of estimates
-1.046	0.979
1.307	2.693
-0.141	0.199
0.086	0.215

Stages run 1200 Error = 0.7649

Std. dev. of error= 0.5410

Parameter estimates	Std. dev.s of estimates
-0.975	0.739
0.904	1.045
-0.104	0.074
0.116	0.133

Stages run 1500 Error = 0.7472
Std. dev. of error= 0.3359

Parameter estimates	Std. dev.s of estimates
-1.167	0.695
0.669	0.967
-0.118	0.110
0.144	0.163

Stages run 1800 Error = 0.6676
Std. dev. of error= 0.3198

Parameter estimates	Std. dev.s of estimates
-1.081	0.571
0.786	0.848
-0.104	0.081
0.129	0.114

Stages run 2100 Error = 0.6704
Std. dev. of error= 0.4877

Parameter estimates	Std. dev.s of estimates
-1.168	0.796
0.620	0.925
-0.107	0.066
0.140	0.129

Stages run 2400 Error = 0.6177
Std. dev. of error= 0.2320

Parameter estimates	Std. dev.s of estimates
-1.184	0.604
0.611	0.745
-0.103	0.056
0.144	0.116

Stages run 2700 Error = 0.5318
Std. dev. of error= 0.2000

Parameter estimates	Std. dev.s of estimates
-1.215	0.514
0.530	0.652
-0.096	0.059
0.145	0.097

```
Stages run 3000  Error       =  0.4738
      Std. dev. of error=  0.2026
```

Parameter estimates	Std. dev.s of estimates
-1.235	0.499
0.580	0.558
-0.100	0.055
0.141	0.087

```
System order n =  2

The system parameters are(in the ordera1,..,an,b1,..,bn) :
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000

Nominal noise:         Gaussian        Mean= 0.00          Std. dev.= 1.00

Contaminating noise: Laplacian         Mean= 0.00          Std. dev.=10.00

Contamination level =    0.03

The Hampel parameters are:            s1 = 1.00            s2 =-0.27
                                      d1 = 0.12e 01        d2 = 0.35e 01
                                          ucutof = 0.79e 01


Stages run  300  Error    =  1.2634
       Std. dev. of error=  0.9080

Parameter    Std. dev.s of
estimates      estimates
  -0.231        1.366
   0.406        1.489
  -0.090        0.132
   0.051        0.242


Stages run  600  Error    =  1.1772
       Std. dev. of error=  0.9690

Parameter    Std. dev.s of
estimates      estimates
  -0.475        1.331
   0.393        1.611
  -0.075        0.210
   0.130        0.244


Stages run  900  Error    =  0.7411
       Std. dev. of error=  0.3114

Parameter    Std. dev.s of
estimates      estimates
  -0.869        0.628
   0.742        0.838
  -0.109        0.098
   0.106        0.133


Stages run 1200  Error    =  0.6471
```

```
        Std. dev. of error=  0.5154

Parameter     Std. dev.s of
estimates       estimates
  -1.181          0.644
   0.836          0.983
  -0.113          0.096
   0.123          0.134


Stages run 1500  Error    =  0.6653
        Std. dev. of error=  0.3295

Parameter     Std. dev.s of
estimates       estimates
  -1.186          0.679
   0.657          0.843
  -0.114          0.063
   0.144          0.128


Stages run 1800  Error    =  0.6234
        Std. dev. of error=  0.3359

Parameter     Std. dev.s of
estimates       estimates
  -1.189          0.649
   0.792          0.764
  -0.101          0.066
   0.137          0.083


Stages run 2100  Error    =  0.5483
        Std. dev. of error=  0.3651

Parameter     Std. dev.s of
estimates       estimates
  -1.112          0.558
   0.672          0.743
  -0.101          0.053
   0.126          0.090


Stages run 2400  Error    =  0.5360
        Std. dev. of error=  0.2712

Parameter     Std. dev.s of
estimates       estimates
  -1.202          0.546
   0.622          0.677
  -0.105          0.057
   0.138          0.096


Stages run 2700  Error    =  0.4694
        Std. dev. of error=  0.2425
```

Parameter estimates	Std. dev.s of estimates
-1.251	0.487
0.526	0.606
-0.097	0.054
0.145	0.085

Stages run 3000 Error = 0.4612
Std. dev. of error= 0.2068

Parameter estimates	Std. dev.s of estimates
-1.262	0.500
0.596	0.539
-0.114	0.057
0.130	0.102

Jun 12 09:33 noise9 Page 1

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :

-1.425
0.496
0.000
-0.102
0.173
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Laplacian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.03

The Hampel parameters are: s1 = 1.00 s2 =-0.50
d1 = 0.25e 01 d2 = 0.45e 01
ucutof = 0.95e 01

Stages run 300 Error = 1.2220
Std. dev. of error= 0.8375

Parameter estimates	Std. dev.s of estimates
-0.284	1.465
0.180	1.202
-0.107	0.122
0.046	0.251

Stages run 600 Error = 1.5326
Std. dev. of error= 1.5674

Parameter estimates	Std. dev.s of estimates
-0.842	1.857
1.256	2.588
-0.101	0.119
0.096	0.221

Stages run 900 Error = 0.8169
Std. dev. of error= 0.3452

Parameter estimates	Std. dev.s of estimates
-0.800	0.615
0.880	0.933
-0.089	0.115
0.110	0.132

Stages run 1200 Error = 1.1325

```
        Std. dev. of error=  1.4841

Parameter    Std. dev.s of
estimates     estimates
  -0.511        2.187
   0.908        1.476
  -0.105        0.075
   0.058        0.299


Stages run 1500  Error    =  0.7914
        Std. dev. of error=  0.4674

Parameter    Std. dev.s of
estimates     estimates
  -0.975        0.918
   0.778        0.895
  -0.107        0.074
   0.124        0.157


Stages run 1800  Error    =  0.7409
        Std. dev. of error=  0.4808

Parameter    Std. dev.s of
estimates     estimates
  -0.977        0.895
   0.921        0.768
  -0.128        0.075
   0.086        0.146


Stages run 2100  Error    =  0.6807
        Std. dev. of error=  0.4039

Parameter    Std. dev.s of
estimates     estimates
  -1.136        0.829
   0.790        0.760
  -0.098        0.061
   0.123        0.101


Stages run 2400  Error    =  0.5881
        Std. dev. of error=  0.2536

Parameter    Std. dev.s of
estimates     estimates
  -1.209        0.553
   0.685        0.740
  -0.098        0.058
   0.143        0.104


Stages run 2700  Error    =  0.5252
        Std. dev. of error=  0.2799
```

Parameter estimates	Std. dev.s of estimates
-1.280	0.564
0.571	0.678
-0.102	0.060
0.145	0.107

Stages run 3000 Error = 0.5372
Std. dev. of error= 0.2311

Parameter estimates	Std. dev.s of estimates
-1.249	0.542
0.650	0.654
-0.111	0.049
0.134	0.109

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000

Nominal noise:         Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise:   Gaussian        Mean= 0.00        Std. dev.=10.00

Contamination level =     0.10

The Hampel parameters are:            s1 = 1.00           s2 = 1.00
                                      d1 = 0.17e 38       d2 = 0.17e 38
                                           ucutof = 0.17e 38


Stages run  300  Error    =  1.4392
        Std. dev. of error=  0.8631

Parameter     Std. dev.s of
estimates       estimates
  -0.125          1.417
   1.083          1.419
  -0.240          0.311
   0.101          0.592


Stages run  600  Error    =  2.5137
        Std. dev. of error=  2.9001

Parameter     Std. dev.s of
estimates       estimates
  -0.893          3.265
   1.321          4.577
  -0.106          0.273
   0.244          1.209


Stages run  900  Error    =  1.8827
        Std. dev. of error=  2.2558

Parameter     Std. dev.s of
estimates       estimates
   0.569          3.490
   0.727          1.713
  -0.099          0.228
  -0.082          0.862


Stages run 1200  Error    =  1.4338
```

Std. dev. of error= 0.9163

Parameter estimates	Std. dev.s of estimates
-0.225	1.433
0.856	1.704
-0.111	0.227
0.059	0.344

Stages run 1500 Error = 1.2922
Std. dev. of error= 1.2109

Parameter estimates	Std. dev.s of estimates
-0.162	1.325
0.053	1.663
-0.205	0.664
-0.006	0.687

Stages run 1800 Error = 1.2868
Std. dev. of error= 1.2070

Parameter estimates	Std. dev.s of estimates
-0.533	1.337
0.750	1.858
-0.229	0.719
0.100	0.758

Stages run 2100 Error = 1.0818
Std. dev. of error= 0.7483

Parameter estimates	Std. dev.s of estimates
-0.434	0.964
0.762	1.396
-0.129	0.148
0.137	0.235

Stages run 2400 Error = 1.3361
Std. dev. of error= 1.5847

Parameter estimates	Std. dev.s of estimates
-0.320	1.349
0.309	2.607
-0.105	0.170
0.156	0.233

Stages run 2700 Error = 1.4829
Std. dev. of error= 1.4044

Parameter estimates	Std. dev.s of estimates
-0.780	2.088
1.011	2.115
-0.116	0.160
0.182	0.352

Stages run 3000 Error = 1.4060
Std. dev. of error= 1.7378

Parameter estimates	Std. dev.s of estimates
-0.119	1.571
0.061	2.670
-0.091	0.178
0.144	0.227

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :

```
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000
```

Nominal noise:	Gaussian	Mean= 0.00	Std. dev.= 1.00
Contaminating noise:	Gaussian	Mean= 0.00	Std. dev.=10.00

Contamination level = 0.10

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.20e 01 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 1.9327
Std. dev. of error= 3.2329

Parameter estimates	Std. dev.s of estimates
-0.791	2.406
-0.408	5.070
-0.134	0.189
0.030	0.324

Stages run 600 Error = 1.6729
Std. dev. of error= 1.7616

Parameter estimates	Std. dev.s of estimates
-0.045	1.350
1.247	3.031
-0.103	0.149
-0.031	0.363

Stages run 900 Error = 7.8522
Std. dev. of error= 35.8222

Parameter estimates	Std. dev.s of estimates
1.407	9.563
10.920	53.908
0.074	1.026
0.219	0.870

Stages run 1200 Error = 1.2513

```
        Std. dev. of error=  1.0506

Parameter    Std. dev.s of
estimates      estimates
  -0.515         1.344
   0.434         1.866
  -0.116         0.134
   0.105         0.200


Stages run 1500  Error    =  1.1218
        Std. dev. of error=  1.0439

Parameter    Std. dev.s of
estimates      estimates
  -0.823         1.641
   0.687         1.510
  -0.113         0.127
   0.150         0.225


Stages run 1800  Error    =  0.8705
        Std. dev. of error=  0.8076

Parameter    Std. dev.s of
estimates      estimates
  -0.846         0.834
   1.029         1.389
  -0.118         0.082
   0.144         0.118


Stages run 2100  Error    =  0.8271
        Std. dev. of error=  0.6764

Parameter    Std. dev.s of
estimates      estimates
  -0.805         0.700
   0.974         1.235
  -0.106         0.083
   0.131         0.090


Stages run 2400  Error    =  0.7951
        Std. dev. of error=  0.7122

Parameter    Std. dev.s of
estimates      estimates
  -1.031         1.141
   0.802         1.023
  -0.101         0.155
   0.151         0.134


Stages run 2700  Error    =  0.7719
        Std. dev. of error=  0.7994
```

Parameter estimates	Std. dev.s of estimates
-0.701	0.904
0.429	1.220
-0.087	0.124
0.129	0.104

```
Stages run 3000  Error    =  0.5953
       Std. dev. of error=  0.4421
```

Parameter estimates	Std. dev.s of estimates
-0.860	0.610
0.409	0.736
-0.092	0.131
0.135	0.120

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
-1.425
0.496
0.000
-0.102
0.173
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Gaussian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.10

The Hampel parameters are: s1 = 1.00 s2 = 0.00
d1 = 0.15e 01 d2 = 0.17e 38
ucutof = 0.17e 38

Stages run 300 Error = 1.8582
Std. dev. of error= 2.2992

Parameter estimates	Std. dev.s of estimates
-0.396	2.529
0.819	3.545
-0.118	0.171
0.096	0.293

Stages run 600 Error = 1.3361
Std. dev. of error= 0.9530

Parameter estimates	Std. dev.s of estimates
-0.406	1.015
0.956	1.965
-0.142	0.150
0.032	0.239

Stages run 900 Error = 1.2934
Std. dev. of error= 0.7963

Parameter estimates	Std. dev.s of estimates
-0.552	0.985
0.995	1.819
-0.096	0.134
0.077	0.168

Stages run 1200 Error = 1.2339

```
      Std. dev. of error=  0.9876

Parameter    Std. dev.s of
estimates      estimates
  -0.663         1.339
   0.467         1.829
  -0.122         0.141
   0.106         0.203


Stages run 1500  Error   =  1.0724
      Std. dev. of error=  1.1372

Parameter    Std. dev.s of
estimates      estimates
  -0.720         1.047
   0.708         1.995
  -0.104         0.115
   0.135         0.168


Stages run 1800  Error   =  0.8689
      Std. dev. of error=  0.9477

Parameter    Std. dev.s of
estimates      estimates
  -1.086         1.085
   0.798         1.555
  -0.103         0.102
   0.135         0.121


Stages run 2100  Error   =  0.8348
      Std. dev. of error=  1.0461

Parameter    Std. dev.s of
estimates      estimates
  -0.872         0.563
   1.043         1.792
  -0.113         0.062
   0.122         0.103


Stages run 2400  Error   =  0.7068
      Std. dev. of error=  0.6790

Parameter    Std. dev.s of
estimates      estimates
  -0.946         0.574
   0.847         1.233
  -0.099         0.102
   0.134         0.100


Stages run 2700  Error   =  0.7135
      Std. dev. of error=  0.9047
```

Parameter estimates	Std. dev.s of estimates
-0.948	0.670
0.348	1.536
-0.105	0.075
0.131	0.095

Stages run 3000 Error = 0.5314
Std. dev. of error= 0.3059

Parameter estimates	Std. dev.s of estimates
-1.111	0.608
0.538	0.596
-0.101	0.162
0.137	0.133

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000

Nominal noise:        Gaussian        Mean= 0.00        Std. dev.= 1.00

Contaminating noise:  Gaussian        Mean= 0.00        Std. dev.=10.00

Contamination level =     0.10

The Hampel parameters are:          s1 = 1.00            s2 = 0.00
                                    d1 = 0.10e 01        d2 = 0.17e 38
                                        ucutof = 0.17e 38


Stages run  300  Error    =  2.7769
       Std. dev. of error=  7.6474

Parameter     Std. dev.s of
estimates       estimates
   1.428         10.316
   1.633          5.923
  -0.182          0.281
  -0.250          1.474


Stages run  600  Error    =  1.2755
       Std. dev. of error=  1.1086

Parameter     Std. dev.s of
estimates       estimates
  -0.502          1.245
   0.926          1.989
  -0.115          0.142
   0.070          0.206


Stages run  900  Error    =  1.1136
       Std. dev. of error=  0.8071

Parameter     Std. dev.s of
estimates       estimates
  -0.409          0.808
   0.798          1.600
  -0.092          0.127
   0.074          0.147


Stages run 1200  Error    =  1.0546
```

Std. dev. of error= 1.0074

Parameter estimates	Std. dev.s of estimates
-0.717	0.996
0.982	1.774
-0.115	0.131
0.102	0.171

Stages run 1500 Error = 0.9873
Std. dev. of error= 1.0561

Parameter estimates	Std. dev.s of estimates
-0.790	0.974
0.885	1.815
-0.090	0.132
0.144	0.158

Stages run 1800 Error = 0.7576
Std. dev. of error= 0.7085

Parameter estimates	Std. dev.s of estimates
-0.944	0.597
0.908	1.309
-0.108	0.074
0.138	0.121

Stages run 2100 Error = 0.8180
Std. dev. of error= 1.0616

Parameter estimates	Std. dev.s of estimates
-0.951	0.536
0.995	1.839
-0.103	0.073
0.129	0.108

Stages run 2400 Error = 0.6734
Std. dev. of error= 0.5808

Parameter estimates	Std. dev.s of estimates
-1.010	0.618
0.744	1.092
-0.090	0.108
0.147	0.105

Stages run 2700 Error = 0.7686
Std. dev. of error= 0.7594

Parameter estimates	Std. dev.s of estimates
-1.201	1.148
0.423	1.141
-0.111	0.118
0.151	0.119

Stages run 3000 Error = 0.6136
Std. dev. of error= 0.6644

Parameter estimates	Std. dev.s of estimates
-1.146	0.542
0.685	1.212
-0.106	0.080
0.130	0.103

```
System order n =  2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
  -1.425
   0.496
   0.000
  -0.102
   0.173
   0.000

Nominal noise:         Gaussian         Mean= 0.00        Std. dev.= 1.00

Contaminating noise:   Gaussian         Mean= 0.00        Std. dev.=10.00

Contamination level =      0.10

The Hampel parameters are:           s1 = 1.00                s2 = 0.00
                                     d1 = 0.70e 00            d2 = 0.17e 38
                                          ucutof = 0.17e 38


Stages run  300  Error    =   1.2262
       Std. dev. of error=   0.8646

Parameter     Std. dev.s of
estimates       estimates
  -0.013          1.189
   0.291          1.288
  -0.111          0.153
   0.048          0.255


Stages run  600  Error    =   1.2445
       Std. dev. of error=   0.9071

Parameter     Std. dev.s of
estimates       estimates
  -0.595          1.167
   0.878          1.796
  -0.115          0.128
   0.049          0.195


Stages run  900  Error    =   1.3012
       Std. dev. of error=   1.2795

Parameter     Std. dev.s of
estimates       estimates
  -0.726          1.954
   0.642          1.819
  -0.105          0.111
   0.041          0.221


Stages run 1200  Error    =   1.0301
```

Std. dev. of error= 0.9563

Parameter estimates	Std. dev.s of estimates
-0.814	0.985
0.663	1.775
-0.124	0.127
0.092	0.182

Stages run 1500 Error = 0.9741
Std. dev. of error= 0.8148

Parameter estimates	Std. dev.s of estimates
-0.728	0.989
0.611	1.352
-0.189	0.464
0.050	0.439

Stages run 1800 Error = 0.7746
Std. dev. of error= 0.7524

Parameter estimates	Std. dev.s of estimates
-0.879	0.588
0.871	1.375
-0.091	0.105
0.134	0.115

Stages run 2100 Error = 0.8052
Std. dev. of error= 0.9588

Parameter estimates	Std. dev.s of estimates
-0.855	0.637
0.967	1.627
-0.069	0.141
0.141	0.104

Stages run 2400 Error = 0.7561
Std. dev. of error= 0.7479

Parameter estimates	Std. dev.s of estimates
-1.065	0.697
0.638	1.400
-0.105	0.106
0.131	0.107

Stages run 2700 Error = 0.6503
Std. dev. of error= 0.5245

Parameter estimates	Std. dev.s of estimates
-1.030	0.652
0.630	0.995
-0.087	0.120
0.144	0.114

Stages run 3000 Error = 0.6511
Std. dev. of error= 0.5423

Parameter estimates	Std. dev.s of estimates
-0.976	0.738
0.671	0.932
-0.079	0.093
0.136	0.096

System order n = 2

The system parameters are(in the ordera1,..,an,b1,..,bn) :

-1.425
0.496
0.000
-0.102
0.173
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Gaussian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.10

The Hampel parameters are: s1 = 1.00 s2 =-0.27
d1 = 0.12e 01 d2 = 0.35e 01
ucutof = 0.79e 01

Stages run 300 Error = 1.5421
Std. dev. of error= 1.5504

Parameter estimates	Std. dev.s of estimates
-0.419	1.816
0.430	2.575
-0.131	0.184
0.044	0.291

Stages run 600 Error = 1.2189
Std. dev. of error= 0.7402

Parameter estimates	Std. dev.s of estimates
-0.242	1.069
0.705	1.429
-0.088	0.121
0.051	0.226

Stages run 900 Error = 2.0823
Std. dev. of error= 5.4919

Parameter estimates	Std. dev.s of estimates
-1.950	8.043
0.512	1.645
0.365	2.464
0.542	2.416

Stages run 1200 Error = 1.0178

```
        Std. dev. of error=  0.7459

Parameter     Std. dev.s of
estimates       estimates
  -0.853         1.121
   0.919         1.370
  -0.106         0.116
   0.125         0.185


Stages run 1500  Error    =  0.9892
        Std. dev. of error=  0.8866

Parameter     Std. dev.s of
estimates       estimates
  -0.915         1.304
   0.933         1.369
  -0.107         0.103
   0.152         0.231


Stages run 1800  Error    =  0.7291
        Std. dev. of error=  0.6216

Parameter     Std. dev.s of
estimates       estimates
  -0.975         0.619
   0.799         1.195
  -0.111         0.072
   0.139         0.125


Stages run 2100  Error    =  0.7615
        Std. dev. of error=  0.7639

Parameter     Std. dev.s of
estimates       estimates
  -1.012         0.628
   0.855         1.409
  -0.103         0.065
   0.136         0.104


Stages run 2400  Error    =  0.6415
        Std. dev. of error=  0.4833

Parameter     Std. dev.s of
estimates       estimates
  -1.074         0.627
   0.623         0.972
  -0.103         0.077
   0.142         0.102


Stages run 2700  Error    =  2.8049
        Std. dev. of error= 12.0311
```

Parameter estimates	Std. dev.s of estimates
-0.540	3.037
-2.834	18.138
-0.372	1.432
-0.074	1.154

Stages run 3000 Error = 0.5742
Std. dev. of error= 0.3782

Parameter estimates	Std. dev.s of estimates
-1.118	0.567
0.527	0.814
-0.085	0.088
0.148	0.084

System order n = 2

The system parameters are(in the ordera1,...,an,b1,...,bn) :
-1.425
0.496
0.000
-0.102
0.173
0.000

Nominal noise: Gaussian Mean= 0.00 Std. dev.= 1.00

Contaminating noise: Gaussian Mean= 0.00 Std. dev.=10.00

Contamination level = 0.10

The Hampel parameters are: s1 = 1.00 s2 =-0.50
d1 = 0.25e 01 d2 = 0.45e 01
ucutof = 0.95e 01

Stages run 300 Error = 1.8963
Std. dev. of error= 2.4334

Parameter estimates	Std. dev.s of estimates
-0.948	2.394
0.314	3.983
-0.159	0.159
0.040	0.392

Stages run 600 Error = 1.7452
Std. dev. of error= 1.9129

Parameter estimates	Std. dev.s of estimates
0.511	2.930
0.731	1.629
-0.095	0.150
-0.096	0.630

Stages run 900 Error = 1.7048
Std. dev. of error= 1.6624

Parameter estimates	Std. dev.s of estimates
-0.316	2.011
1.270	2.675
-0.088	0.159
0.124	0.283

Stages run 1200 Error = 1.7496

Std. dev. of error= 2.2380

Parameter estimates	Std. dev.s of estimates
-0.279	3.102
1.307	2.541
-0.152	0.430
0.031	0.664

Stages run 1500 Error = 1.0392
Std. dev. of error= 0.6749

Parameter estimates	Std. dev.s of estimates
-0.958	1.186
0.844	1.327
-0.139	0.112
0.147	0.186

Stages run 1800 Error = 0.8679
Std. dev. of error= 0.6904

Parameter estimates	Std. dev.s of estimates
-0.790	0.820
0.991	1.226
-0.124	0.084
0.127	0.129

Stages run 2100 Error = 0.9006
Std. dev. of error= 0.7942

Parameter estimates	Std. dev.s of estimates
-0.865	0.816
1.056	1.425
-0.122	0.064
0.115	0.105

Stages run 2400 Error = 0.7024
Std. dev. of error= 0.4960

Parameter estimates	Std. dev.s of estimates
-0.821	0.636
0.775	0.917
-0.097	0.108
0.130	0.107

Stages run 2700 Error = 0.7800
Std. dev. of error= 0.6443

Parameter estimates	Std. dev.s of estimates
-0.897	0.838
0.430	1.169
-0.112	0.080
0.127	0.106

Stages run 3000 Error = 0.7692
Std. dev. of error= 0.8197

Parameter estimates	Std. dev.s of estimates
-1.050	0.827
0.699	1.430
-0.111	0.061
0.114	0.106

APPENDIX A

ROBUST QUANTIZATION OF DISCRETE-TIME SIGNALS WITH UNIMODAL DISTRIBUTIONS AND GENERALIZED MOMENT CONSTRAINTS

I. Introduction

With the continuing evolution of digital processing elements the task of discretizing signals in communication and control systems is becoming increasingly important. Perhaps the most widely used method of converting discrete-time analog data to digital form is a simple amplitude quantizer. Often a uniform quantizer is used due to its simplicity and general good performance[1,2]. However, if the class of input signals can be restricted by a priori information, the designer should be able to improve performance by constructing a quantizer matched to the characteristics of the input signal. Typically, the a priori information is the specification of the cumulative probability distribution of the input signal. However, in many cases the distribution of the input signal is only approximately known, so that the design task is complicated. In these cases a robust quantizer may provide the most acceptable performance. The robust quantizer will guarantee the best possible performance commensurate with the available knowledge of the input distribution.

This paper considers robust quantization when the a priori information consists of knowledge that the input signal distribution is unimodal and satisfies a generalized moment constraint. The restriction to unimodal distributions is a natural one and results in least-favorable distributions much more realistic than those found in similar problems[3,4]. The generalized moment condition includes the possibility of using standard moments, such as variance, as well as approximations to other forms, such as percentile constraints. With these constraints defining the admissible distributions and with performance measured by a mean weighted quantization error criterion, the problem is to find the least-favorable distribution and the best N-level quantizer in the minimax sense. Section II of the paper

presents the formal statement of the problem. In section III the existence of a solution is demonstrated, necessary conditions are presented and an algorithm for finding the solution is given. Section IV considers the performance of the robust quantizer for a specific choice of the error weighting function and moment constraint.

II. Problem Statement and Preliminaries

Let X denote the real random variable to be digitized and let F denote any of the possible cumulative distribution functions of X. The distribution, F, is restricted by the generalized moment constraint,

$$G(F) = \int_{-\infty}^{\infty} \rho(s)dF(s) \leq c, \tag{1}$$

where ρ is a convex, monotonically increasing function of $|s|$. The most typical choice would be a moment, $\rho(s) = |s|^p$. We also impose the condition that the distribution be unimodal as defined by Feller[5]:

> Definition: A distribution function F is called unimodal with mode at the origin if and only if the graph of F is convex in $[-\infty, 0)$ and concave in $(0,\infty]$. The origin may be a point of discontinuity, but apart from this, unimodality requires that there exist a density f which is monotone in $[-\infty,0)$ and in $(0,\infty]$. (Intervals of constancy are not excluded.)

Denote the set of distributions satisfying these two constraints by D, which we take to be a set in the normalized space of functions of bounded variation on the extended real line ($NBV[-\infty,\infty]$). Since D contains only probability distribution functions, it is a subset of the unit ball in the space, which is compact in the weak* topology by the Banach-Alaoglu theorem[6]. It is easily verified that D the admissible set of distributions is weak* closed. Let D^1 be the set of distributions satisfying (1), and D^2 be the set of unimodal distributions.

<u>Lemma 1</u>: D^1 is weak* closed.

<u>Proof</u>: Since ρ is convex and bounded from below, there exists an increasing sequence ρ_i of bounded continuous functions converging pointwise to ρ. It follows from the definition of the weak* topology that the sets D_i^1 consisting of the distributions satisfying (1) with ρ_i instead of ρ are weak* closed. Also, by the monotone convergence theorem the intersection of the D_i^1 must be the same as D. Thus, by the infinite intersection property, D is weak* closed.

<u>Lemma 2</u>: D^2 is weak* closed.

<u>Proof</u>: Suppose F_i is a sequence of unimodal distributions weak* convergent to a distribution F. We must show that F is unimodal. Since the F_i are continuous, except possibly at the origin, weak* convergence is equivalent to pointwise convergence. Each F_i is convex for negative argument and concave for positive argument; accordingly, the pointwise limit of the functions must similarly be convex and concave -- thus unimodal.

The intersection of two closed sets is closed, so that D is weak* closed. Thus, being a subset of the unit ball, D is also

weak* compact. Furthermore, since G is linear in F and since convex combinations of unimodal distributions are unimodal distributions, D is also a convex set.

> Remark: In order to guarantee existence of solutions to an optimization problem the admissible set must usually be compact. Since compactness with respect to the norm topology is a very severe requirement in a function space, most sets of interest will not be norm compact. Since Weak* compactness is much less stringent, it is a more useful starting point for proving existence of solutions.

A N-level quantizer q is a device with input X and output Y defined by

$$q(x) = \begin{cases} y_1 & x \in [-\infty, b_1) \\ y_i & x \in [b_{i-1}, b_i) \\ y_N & x \in [b_i, \infty] \end{cases} \qquad (2)$$

with $b_0 \leq y_1 \leq y_2 \leq \cdots \leq y_N \leq b_n$ (see below for the definition of b_0 and b_N). The b_i have been called transition values and the y_i representation values. The admissible values of the 2N-1 quantizer parameters clearly define a convex, compact set in R^{2N-1}. Call this set Q.

A number is associated with each quantizer, q, and each distribution, F, which measures the fidelity of the quantizer's representation of the input signal. This number is defined by

$$E(q,F) = \int_{-\infty}^{\infty} g^*(s)dF(s) \quad , \tag{3}$$

where the integrand $g^*(s)$ is an upper semicontinuous function defined by

$$g^*(s) = \begin{cases} g(b_0-y_1) & s\epsilon[-\infty,b_0] \\ g(s-y_i) & s\epsilon(b_{i-1},b_i) \quad i=1,\ldots,N \\ g(b_N-y_N) & s\epsilon[b_N,\infty] \\ \max[g(b_i-y_i),g(b_i-y_{i+1})] & s=b_i \quad i=1,\ldots,N-1 \end{cases} \quad , \tag{4}$$

with $g(t)$ a convex weighting function, monotonic in $|t|$. The numbers b_0 and b_N define the input dynamic range of interest. The assumption being, if the signal exceeds these values no added penalty should be assessed for the increased representation error. The function $E(q,F)$, considered as a functional on D, can be shown to be weak* upper semi-continuous. Considered as a functional on Q it is continuous. However, while $E(q,F)$ is concave (linear) in F, no similar statement can be made with respect to q.

> Remark: The saturation of the weighting function is included for both aesthetic and mathematical reasons. If the rate of increase of g is greater than that of ρ, the problem does not have a solution. If they are the same, then the cost is equal to c, the value of the constraint.

Armed with the above definitions the statement of the problem is simple:

$$\inf_{q \in Q} \sup_{F \in D} E(q,F). \tag{5}$$

III. Main Results

Before attempting to find explicit solutions to the problem, we verify the existence of least-favorable distributions and robust quantizers, so that our search shall not be in vain.

Theorem: For an arbitrary quantizer q in Q there exists a maximizing distribution function in D. Also, there exists a minimizing quantizer for the resulting minimization problem.

Proof: An upper semicontinuous function achieves its supremum on a compact set[6]. Since $E(q,F)$ is weak* upper semicontinuous and D is weak* compact, the conditions are satisfied for any quantizer. The second part of the proposition follows since Q is compact and the sup $E(q,F)$ is lower semicontinuous on Q.

As usual, finding solutions to a problem is a bit more difficult than proving their existence. In this problem, knowing that there exists a distribution solving the problem is of some help in itself for, as Max[7] showed, when finding the optimum quantizer for a given distribution function, the optimal quantizer parameters satisfy

$$b_i = (y_i + y_{i+1})/2 \qquad i=1,2,\ldots,N-1 \tag{6}$$

no matter what the distribution. This same set of equations will hold for our problem; thus, reducing the number of unknown quantizer parameters by almost one-half. The primary tool which will be used to provide necessary conditions that the remaining

quantizer parameters must satisfy is the Lagrange Duality theorem[7]. This theorem is stated below in a form appropriate to the current problem.

Theorem 2(Lagrange Duality): Let C be the convex cone defined by D. Suppose that (q_0,F_0) is a solution to the robust quantization problem and $u_0=E(q_0,F_0)$ is the minimax value of the fidelity criterion. Then the following equation holds:

$$u_0 = \min_{q\in Q} \min_{\lambda\geq 0} \max_{F\in C} \left[E(q,F)-\lambda_1\int_{-\infty}^{\infty} dF(s)-\lambda_2 G(F)+\lambda_1+\lambda_2 c \right], \tag{7}$$

where the multiplier, $\lambda=(\lambda_1,\lambda_2)$. Call the minimizing value of λ, λ^0. Either $\lambda_i^0=0$ or the corresponding constraint is satisfied with equality, that is,

$$\lambda_1^0(G(F_0)-c) = \lambda_2^0\left(\int_{-\infty}^{\infty} dF_0(s)-1\right) = 0. \tag{8}$$

Note that the problem as stated is symmetric about the orgin and there will be no loss in generality by considering only the non-negative half-line as the domain of definition for the problem. We need only set $b_{N/2}$ or $y_{(N+1)/2}$ equal to zero depending on whether N is even or odd, respectively. Also, renumber the parameters so that the b or y set to zero is now subscripted by zero and the new value for N is the old one divided by two or one plus the old value divided by two as appropriate. Having done this, consider the maximization with respect to F in (7). We can write the terms which are a function of F as,

$$\max_{F \in C} \int_0^{\infty} [g^*(s) - \lambda_1 - \lambda_2 \rho(s)] dF(s). \tag{9}$$

Since F is restricted to be concave, it can increase at most linearly. Thus, a necessary and sufficient condition for (9) to be finite is that

$$H(x) = \int_0^{x} [g^*(s) - \lambda_1 - \lambda_2 \rho(s)] ds \leq 0, \quad \forall\ x \geq 0. \tag{10}$$

If this condition holds, then the maximum value of (9) is zero. Using (10) an equivalent statement of the problem is

$$\min_{\lambda \geq 0} \min_{q \in Q} (\lambda_1 + \lambda_2 c), \tag{11}$$

subject to the condition given by (10). We must be able to determine if, for a given λ, there exists a set of quantizer parameters making (10) hold. If we can do this, the problem is reduced to a two dimensional optimization problem, and there are any number of algorithms that can be used to solve it.

Inequality (10) will be satisfied if and only if all of the local maxima of $H(x)$ are less than or equal to zero. A necessary condition of the local extrema of H is

$$g^*(x) - \lambda_1 - \lambda_2 \rho(x) = 0. \tag{12}$$

Because of the form of g^*, this equation can have at most $2N+1$ roots, N of which are local minima or inflection points and $N+1$ which are candidates for local maxima (They may also be inflection points.). The equations for these $N+1$ points, x_i are:

$$g(y_i-x_i) - \lambda_1 - \lambda_2\rho(x_i) = 0 \quad b_i \leq x_i \leq y_i \quad i=1,2,\ldots,N \tag{13}$$

and

$$g(y_N-b_N) - \lambda_1 - \lambda_2\rho(x_{N+1}) = 0 \quad b_N \leq x_{N+1}. \tag{14}$$

> Remark: If the number of quantization levels in the original problem was odd, then $y_1=0$ and (13) holds for $i=2,3,\ldots,N$.

We need to determine if there exist quantizers such that all of the local maxima of H are less than or equal to zero. Consider the construction, described below, for generating a set of quantizer parameters:

1) Starting with y_1 and continuing iteratively through the i's, pick the values of the y_i's so that the value of H(x) is exactly zero at the points x_i, $i=1,2,\ldots,N$.

2) If $H(x_{N+1}) \leq 0$, the constructed quantizer satisfies (10). Alternatively, if $H(x_{N+1}) > 0$, then the construction procedure fails.

This procedure will always find a quantizer satisfying (10) if one exists. We give the following lemma to make this point precise.

Lemma 3: For a given value of λ, if the set of quantizers satisfying inequality (10) is not void, the construction technique described above will generate a quantizer in the set.

The proof of this lemma is not difficult, but is quite messy. Therefore, we will provide only an outline of the proof First, observe that no matter what values λ_1 and λ_2 take, we can always find values for the y_i making part one of the construction true. Thus, we only need prove that no other choice of the y_i will make the second part true if our choice fails. The approach taken is to compute the variation of the N+1 maximum with respect to changes in the y_i, noting that the y's can only be moved to the left and that moving any one of them left implies some motion to the left of all of the others of higher index. The variation shows that any such motion will cause the value of the N+1 maximum to increase, thus continuing to be greater than zero.

By using the construction procedure, the computer program necessary to compute the quantizer parameters and the minimax value of the error is quite short and requires a minimal amount of central processor time.

IV. Example

To illustrate the behavior of solutions to the problem, we consider a simple example. Let the functions g and ρ both be quadratic and let the total number of quantization intervals be four with $-b_0=b_4=1$. The values of the quantizer parameters and the worst case error were determined for several values of the moment constraint. These are tabulated in table 1 for various values of ratio between the saturation amplitude, b_n and the standard deviation of the signal constraint. Also given are values for the worst case error for the cases from [1] and [2] in which the distribution was not required to be unimodal.

	Unimodal				Not Unimodal			
s.d.	y_1	b_1	y_2	error	y_1	b_1	y_2	error
1.0	0.17	0.49	0.81	.0303	0.25	0.50	0.75	.0625
2/3	0.17	0.48	0.80	.0300	0.25	0.50	0.75	.0625
0.5	0.16	0.47	0.77	.0293	0.20	0.43	0.67	.0589
1/3	0.13	0.38	0.62	.0265	0.12	0.32	0.50	.0403
0.2	.086	0.25	0.41	.0165	.070	0.23	0.38	.0212
0.1	.040	0.13	0.22	.0066	.031	0.14	0.25	.0066

Table 1. Robust Quantizer Performance

References

[1] Goblick, T. J., "Analog Source Digitization: A Comparison of Theory and Practice," IEEE Trans. on Information Theory, vol. IT-13, pp. 323-326, April, 1967.

[2] Max, J., "Quantizing for Minimum Distortion," IRE Trans. on Information Theory, vol. IT-6, pp. 7-12, March, 1960.

[3] Morris, J. M. and VandeLinde, V. D., "Robust Quantization of Discrete-Time Signals with Independent Samples," IEEE Trans. on Communications, vol. COMM-22, pp. 1897-1902, December, 1974.

[4] VandeLinde, V. D., "Robust Quantization with Generalized Moment Constraints," Proceedings of the Thirteenth Annual Allerton Conference on Circuit and System Theory, pp. 774-780, October, 1975.

[5] Feller, W., "An Introduction to Probability Theory and its Applications, Vol. II," p 155, New York: Wiley, 1966.

[6] Ash, R. B., "Real Analysis and Probability," New York: Academic Press, 1972.

[7] Luenberger, D. G., "Optimization by Vector Space Methods," New York: Wiley, 1969.

APPENDIX B

ROBUST PROPERTIES OF SOLUTIONS TO LINEAR-QUADRATIC ESTIMATION AND CONTROL PROBLEMS

In this note we hope to provide insight into the optimality properties of the standard solutions to a large class of linear-quadratic estimation and control problems, when they are viewed in a more general context. Of importance to the systems engineer when he designs an estimation and control system is the sensitivity of that design to deviations from the assumed model of the environment. Sensitivity analysis of plant parameters is an extensively studied discipline for linear systems; but, beyond parametic sensitivity studies of the effect of changes in the first two moments of the noise densities, little work has been devoted to the sensitivity of such systems to deviations from the assumed statistical model. However, in the statistical literature this problem has received a great deal of attention in the last decade[1,2,3], leading to a subarea of statistics which we refer to here as robust design. Loosely speaking, a robust policy may be described as one which performs well even though the actual state of nature deviates "mildly" from the environment nominally assumed in the design. We put the word mildly in quotation marks since it is a key word in the entire concept. Quantifying mildly is impossible without a good physical feel for the noise generating process and its inherent constraints. If, for example, the noise is generated directly from a well understood and accurately described process, such as thermal noise in a resistor, then tight bounds may be placed on the statistical description of the noise process, and a mild deviation is a very small change in the process statistics. If, on the other hand, the noise to be modeled is a process derived from a source which is not very well understood or quantified, then the nominal statistical description may be imprecise. Thus, mild deviations from the nominal description may represent large changes in the process statistics.

The hope of the engineer who adheres to the philosophy of robust design is that he may be able to find a policy performing

within a few percent of optimality for the most likely environment while protecting against performance deterioration of much greater magnitude caused by "mild" deviations from the assumed environment. The most frequent formalization of the robust design problem is as a mathematical game between the engineer and nature. How well the robust design can be effected will depend on the situation at hand and the engineer's ability to pose the most appropriate game to be solved. A solution to the resulting minimax optimization problem consists of a most robust policy and a least favorable state for nature. If the least favorable state happens to correspond to the most likely or nominal state, the solution is particularly meaningful, since no penalty need be paid for robustness. As we shall show, this is the situation in many linear-quadratic estimation and control problems. First, however, we need some basic concepts from game theory.

An abstract game may be defined as a triple (A,B,V), where A and B are the sets of possible strategies for the engineer and for nature, respectively, and V is a function from AxB to the real line which measures the performance of a strategy pair (a,b). Nature chooses b to maximize V, and the engineer chooses a to minimize it. A pair (a_0, b_0) is called a saddlepoint pair for the game if $V(a_0, b_0)$, the value of the game satisfies

$$V(a_0, b) \leq V(a_0, b_0) \quad \forall \quad b \in B \tag{1}$$

and

$$V(a_0, b_0) \leq V(a, b_0) \quad \forall \quad a \in A. \tag{2}$$

It is not always possible to find a pair satisfying (1) and (2), and in such cases more extensive considerations are necessary. For the problems considered here a saddlepoint will exist; thus,

there is no need to discuss the possible additional complications (the interested reader should see Ferguson[4]).

This setup can be applied directly to many linear-quadratic estimation and control problems. Let A contain all functions, both linear and nonlinear, of the observations and let B contain all probability distributions satisfying the standard first and second moment constraints. Futhermore, let a_0 correspond to the best linear solution to the problem and let b_0 correspond to the Gaussian distribution function satisfying the moment constraints (with greatest covariance matrix, if the second moment constraint is an inequality). Then, inequality (1) requires that the linear solution perform no more poorly with respect to any other distribution satisfying the constraints than it does with respect to the Gaussian. Since the problem is linear and the performance measure quadratic, a linear solution yields a value for V that is only a function of at most the first two moments of the noise distribution. These are constrained by hypothesis, so that (1) holds with equality. Inequality (2) is a statement that the linear solution must be optimal with respect to Gaussian distributions for the noise. Thus, if any model admits the same solution to both the linear-least-squares problem and the optimal Gaussian problem, then this same solution also solves the minimax-robust problem. A large class of linear-quadratic estimation and control problems have solutions that satisfy this condition. The ease with which this fact is obtained should not belittle its importance. The philosophical ramifications of this added property are profound for, with it, we can provide added justification for using linear policies even when the noise is suspected to be non-Gaussian. Also illuminated is the extreme importance of the constraint on the second moment of the noise. It is this constraint on which the efficacy of the linear solution rests and not on the Gaussian assumption. As we have seen, any arbitrary deviation from Gaussian noise, which

continues to satisfy the second moment constraint, cannot increase the cost. However, there exist distributions looking, for all the world, like the Gaussian which can drive the cost arbitrarily high; for example, consider a mixture distribution with the Gaussian being the true distribution 99.9% of the time and a Cauchy the true distribution 0.1% of the time. To detect the fact that the variance of the distribution is very large (in fact infinite) would require many thousands of observations; however, any given sample function of the noise could drive the estimation error or control cost very high.

Although the above results have escaped the general knowledge of the engineering community, they are not entirely without precedent. In fact, for the Kalman filtering problem the robust property of the solution is almost as old as the filter itself, having been pointed out by Carlton[5]. In various specific problems[6,7] other workers have used similar ideas in a limited context. We believe appreciation of the simple result reported in this note and an application of the ideas in a broader research context can be of major importance as a direction for research in the next few years.

References

[1] Huber, Peter J., "Robust Estimation of a Location Paramter," Ann. Math. Statist., Vol. 34, pp. 73-101, March, 1964.

[2] Gastwirth, Joesph L., "On Robust Procedures," J. Am. Stat. Assoc., Vol. 61, pp. 929-948, 1966.

[3] Hampel F. R., "A General Qualitative Definition of Robustness," Ann. Math. Statist., Vol. 41, pp. 1887-1896. Dec., 1971.

[4] Ferguson, Thomas S., "Mathematical Statistics," New York: Academic Press, 1967.

[5] Carlton, A. George, "Linear Estimation in Stochastic Processes," The Johns Hopkins University Applied Physics Laboratory, Bumblebee series report #311, March, 1962, pp 72-73.

[6] Basar, T. and Mintz, M., "On a Minimax Estimate for the Mean of a Normal Random Vector Under a Generalized Quadratic Loss Function," Ann. of Statist, Vol. 1, #1, 1973.

[7] Mintz, M., "A Kalman Filter as a Minimax Estimator," J. of Opt. Theory and Appl., Vol. 9, #2, 1972.

DISTRIBUTION LIST

No. of Copies	Organization
12	Commander Defense Documentation Center ATTN: DDC-TCA Cameron Station Alexandria, VA 22314
1	Commander US Army Materiel Development and Readiness Command ATTN: DRCDMA-ST 5001 Eisenhower Avenue Alexandria, VA 22333
1	Commander US Army Aviation Systems Command ATTN: DRSAV-E 12th and Spruce Streets St. Louis, MO 63166
1	Director US Army Air Mobility Research and Development Laboratory Ames Research Center Moffett Field, CA 94035
1	Commander US Army Electronics Command ATTN: DRSEL-RD Fort Monmouth, NJ 07703
1	Commander US Army Missile Command ATTN: DRSMI-R Redstone Arsenal, AL 35809
1	Commander US Army Tank Automotive Development Command ATTN: DRDTA-RWL Warren, MI 48090
2	Commander US Army Mobility Equipment Research & Development Command ATTN: Tech Docu Cen, Bldg. 315 DRSME-RZT Fort Belvoir, VA 22060
1	Commander US Army Armament Command Rock Island, IL 61202
1	Commander US Army Frankford Arsenal ATTN: SARFA-FCV, Stan Goodman Philadelphia, PA 19137
1	Commander US Army Harry Diamond Labs ATTN: DRXDO-TI 2800 Powder Mill Road Adelphi, MD 20783
1	Director US Army TRADOC Systems Analysis Activity ATTN: ATAA-SA White Sands Missile Range NM 88002
1	The Johns Hopkins University ATTN: Dr. W. H. Huggins Dept of Electrical Engineering Baltimore, MD 21218
1	University of Wisconsin Math Research Center 610 Walnut Street Madison, WI 53706

Aberdeen Proving Ground

Marine Corps Ln Ofc
Dir, USAMSAA